THE CONTACT INVARIANT IN SUTURED FLOER HOMOLOGY

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Abstract. We describe an invariant of a contact 3-manifold with convex boundary as an element of Juhász's sutured Floer homology. Our invariant generalizes the contact invariant in Heegaard Floer homology in the closed case, due to Ozsváth and Szabó.

The goal of this paper is to present an invariant of a contact 3-manifold with convex boundary. The contact class is an element of *sutured Floer homology*, defined by András Juhász, and generalizes the contact class in Heegaard Floer homology in the closed case, as defined by Ozsváth and Szabó [OS3] and reformulated by the authors in [HKM2]. In this paper we assume familiarity with convex surface theory (cf. [Gi1, H1]), Heegaard Floer homology (cf. [OS1, OS2]), and sutured manifold theory (cf. [Ga]).

Sutured manifold theory was introduced by Gabai [Ga] to study and construct taut foliations. A *sutured manifold* (M, Γ) is a compact oriented 3-manifold (not necessarily connected) with boundary, together with a compact subsurface $\Gamma = A(\Gamma) \sqcup T(\Gamma) \subset \partial M$, where $A(\Gamma)$ is a union of pairwise disjoint annuli and $T(\Gamma)$ is a union of tori. We orient each component of $\partial M - \Gamma$, subject to the condition that the orientation changes every time we nontrivially cross $A(\Gamma)$. Let $R_+(\Gamma)$ (resp. $R_-(\Gamma)$) be the open subsurface of $\partial M - \Gamma$ on which the orientation agrees with (resp. is the opposite of) the boundary orientation on ∂M. Moreover, Γ is oriented so that the orientation agrees with the orientation of $\partial R_+(\Gamma)$ and is the opposite that of $\partial R_-(\Gamma)$. A sutured manifold (M, Γ) is *balanced* if M has no closed components, $\pi_0(A(\Gamma)) \to \pi_0(\partial M)$ is surjective, and $\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))$ on every component of M. In particular, $\Gamma = A(\Gamma)$ and every boundary component of ∂M nontrivially intersects the suture Γ. *In this paper, all our sutured manifolds are assumed to be balanced.*

In what follows, we view the suture Γ as the union of cores of the annuli, i.e., as a disjoint union of simple closed curves.

In the setting of contact structures, a natural condition to impose on a contact 3-manifold (M, ξ) with boundary is to require that ∂M be *convex*, i.e., there is a contact vector field transverse to ∂M. To a convex surface F one can associate its *dividing set* Γ_F — it is defined as the isotopy class of multicurves $\{x \in F | X(x) \in \xi(x)\}$, where X is a transverse contact vector field. It was explained in [HKM3] that dividing sets and sutures on balanced sutured manifolds can be viewed as equivalent objects. As usual, our contact structures are assumed to be cooriented.

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In a pair of important papers [Ju1, Ju2], András Juhász generalized (the hat versions of) Ozsváth and Szabó's Heegaard Floer homology [OS1, OS2] and link Floer homology [OS4] theories, and assigned a Floer homology group $SFH(M,\Gamma)$ to a balanced sutured manifold (M,Γ). (A similar theory was also worked out by Lipshitz [Li2].) An important property of this *sutured Floer homology* is the following: if $(M,\Gamma) \overset{T}{\rightsquigarrow} (M',\Gamma')$ is a sutured manifold decomposition along a cutting surface T, then $SFH(M',\Gamma')$ is a direct summand of $SFH(M,\Gamma)$.

The main result of this paper is the following:

Theorem 0.1. *Let (M,Γ) be a balanced sutured manifold, and let ξ be a contact structure on M with convex boundary, whose dividing set on ∂M is Γ. Then there exists an invariant $EH(M,\Gamma,\xi)$ of the contact structure which lives in $SFH(-M,-\Gamma)/\{\pm 1\}$.*

Here we are using $\mathbb{Z}$-coefficients. Note that there is currently a ± 1 ambiguity when $\mathbb{Z}$-coefficients are used.

The paper is organized as follows. Section 1 is devoted to discussing the contact-topological preliminaries for obtaining a partial open book decomposition of a contact 3-manifold with convex boundary. We define the contact invariant in Section 2 and prove that it is independent of the choices made in Section 3. We discuss some basic properties of the contact class in Section 4 and compute some examples in Section 5. Finally, we explain the relationship to sutured manifold decompositions in Section 6.

1. Contact structure preliminaries

Let (M,Γ) be a sutured manifold. Let ξ be a contact structure on M with convex boundary so that the dividing set $\Gamma_{\partial M}$ on ∂M is isotopic to Γ. Such a contact manifold will be denoted (M,Γ,ξ).

The following theorem is the key to obtaining a partial open book decomposition, slightly generalizing the work of Giroux [Gi2] to the relative case. For more detailed expositions of Giroux's work, see [Co2, Et].

Theorem 1.1. *There exists a Legendrian graph $K \subset M$ whose endpoints, i.e., univalent vertices, lie on $\Gamma \subset \partial M$ and which satisfies the following:*

1. *There is a neighborhood $N(K) \subset M$ of K so that (i) $\partial N(K) = T \cup (\cup_i D_i)$, (ii) T is a convex surface with Legendrian boundary, (iii) $D_i \subset \partial M$ is a convex disk with Legendrian boundary, (iv) $T \cap \partial M = \cup_i \partial D_i$, (v) $\#(\partial D_i \cap \Gamma_{\partial M}) = 2$, and (vi) there is a system of pairwise disjoint compressing disks D'_j for $N(K)$ so that $\partial D'_j \subset T$, $|\partial D'_j \cap \Gamma_T| = 2$, and each component of $N(K) - \cup_j D'_j$ is a standard contact 3-ball, after rounding the corners.*
2. *Each component H of the complement $M - N(K)$ is a handlebody with convex boundary. There is a system of pairwise disjoint compressing disks D^α_k for H so that $|\partial D^\alpha_k \cap \Gamma_{\partial H}| = 2$ and $H - \cup_k D^\alpha_k$ is a standard contact 3-ball, after rounding the corners.*

Here $|\cdot|$ denotes the geometric intersection number and $\#(\cdot)$ denotes the number of connected components. A *standard contact 3-ball* is a tight contact 3-ball B^3 with convex

boundary and $\#\Gamma_{\partial B^3} = 1$. We say that a handlebody H with convex boundary admits a *product disk decomposition* if condition (2) of Theorem 1.1 holds.

Proof. Since $F = \partial M$ is convex, there is an $I = [0,1]$-invariant contact neighborhood $F \times I \subset M$ so that $F \times \{1\} = \partial M$. First we take a polyhedral decomposition of $F_0 = F \times \{0\}$ so that the 1-skeleton is Legendrian and the boundary of each 2-cell intersects Γ_{F_0} in two points. During this process we need to use the Legendrian realization principle and slightly isotop F_0. Next, extend the polyhedral decomposition on F_0 to $M - (F \times I)$ so that the 1-skeleton K' is Legendrian and the boundary of each 2-cell has Thurston-Bennequin invariant $tb = -1$. Finally, we need to connect K' to ∂M. To achieve this, for each intersection point p of $K'|_{F_0}$ with Γ_{F_0}, we add an edge $\{p\} \times [0,1] \subset F \times [0,1]$ to K'. The resulting graph is our desired K.

We now prove that K satisfies the properties of the theorem. There is a collection of compressing disks in $(M - N(K')) - (F \times I)$ which intersect the dividing set $\Gamma_{\partial N(K')}$ at exactly two points, by the $tb = -1$ condition. Without loss of generality, these compressing disks can be made convex with Legendrian boundary. Then the compressing disks cut up $M - N(K')$ into a disjoint union of standard contact 3-balls and $F \times I$. After removing standard neighborhoods of $\{p\} \times I$ from $F \times I$, the remaining contact 3-manifold admits a product disk decomposition. This implies that $M - N(K)$ admits a product disk decomposition. $\square$

The next theorem is the relative version of the subdivision theorem of Giroux [Gi2] for contact cellular decompositions, which implies that on a closed manifold M any two open books corresponding to a fixed (M, ξ) become isotopic after a sequence of positive stabilizations to each.

Theorem 1.2. *Let K and K' be Legendrian graphs on (M, Γ, ξ) which satisfy the conditions of Theorem 1.1 and $K \cap K' = \emptyset$. Then there exists a common Legendrian extension L of K and K' so that $L = K_n$ is obtained inductively from $K = K_0$ by attaching Legendrian arcs c_i, $i = 1, \dots, n-1$, to the standard neighborhood $N(K_i)$ of K_i so that the following hold:*

1. *The arc c_i has both endpoints on the dividing set of $\partial(M - N(K_i))$, and $int(c_i) \subset int(M - N(K_i))$.*
2. *$N(K_{i+1}) = N(c_i) \cup N(K_i)$, and K_{i+1} is a Legendrian graph so that $N(K_{i+1})$ is its standard neighborhood.*
3. *There is a Legendrian arc d_i on $\partial(M - N(K_i))$ with the same endpoints as c_i, after possible application of the Legendrian realization principle. The arc d_i intersects $\Gamma_{\partial(M - N(K_i))}$ only at its endpoints.*
4. *The Legendrian knot $\gamma_i = c_i \cup d_i$ bounds a disk in $M - N(K_i)$ and has $tb(\gamma_i) = -1$ with respect to this disk. This implies that c_i and d_i are Legendrian isotopic relative to their endpoints inside the closure of $M - N(K_i)$.*

The graph $L = K'_m$ is similarly obtained from $K' = K'_0$ by inductively attaching Legendrian arcs c'_i to $N(K'_i)$ to obtain K'_{i+1}.

Proof. Consider the invariant neighborhood $F \times [0,1]$ with $F = F_1 = \partial M$ as in the first paragraph of Theorem 1.1, and take a copy $F_{1-\delta}$ near F. The procedure for finding a common refinement L of K and K' is as follows:

(1) First add Legendrian arcs to K so that K_i, $i \gg 0$, contains a Legendrian 1-skeleton of the protective layer $F_{1-\delta}$ (as in the first paragraph of Theorem 1.1).
(2) Subdivide the Legendrian 1-skeleton of $F_{1-\delta}$ sufficiently, so that every arc of K' near ∂M (we assume they are all of the form $\{p\} \times [1-\delta, 1]$) intersects the 1-skeleton.
(3) Next add enough Legendrian arcs of the type $\{p\} \times [1-\delta, 1]$, where $p \in \Gamma_F$, so that $K_i \supset (K \cup K') \cap (F \times [1-\delta, 1])$, $i \gg 0$.
(4) Finally, apply the contact subdivision procedure of [Gi2] away from $F \times [1-\delta, 1]$.

Steps (2) and (4) involve the same procedure used in [Gi2], namely, given a face Δ of the contact cellular decomposition with Legendrian boundary $\partial\Delta \subset K$, take a Legendrian arc $c_i \subset \Delta$ with endpoints on $\partial\Delta$, so that c_i cuts Δ into Δ_1, Δ_2, each of which has $tb(\partial\Delta_i) = -1$.

Next we discuss what happens in Steps (1) and (3). There are two types of arcs to attach in Step (1). The first type is an arc $c_i \subset F_{1-\delta}$ from $(p, 1-\delta)$ to $(q, 1-\delta)$, $p, q \in \Gamma_F$, which does not intersect $\Gamma_F \times \{1-\delta\}$ in its interior. Figure 1 depicts this situation. In this case there is

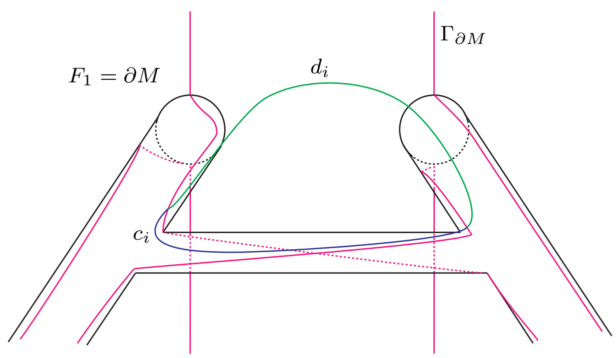

FIGURE 1. Step (1) of the subdivision process. Attaching an arc of the first type. The face $F_1 = \partial M$ is in the back. The cylinders on the left and right are thickenings of $\{p\} \times [0, 1]$ and $\{q\} \times [0, 1]$, and the horizontal cylinder is a thickening of c_i. The blue arc is c_i and the green arc is d_i.

an arc d_i which passes through $R_\pm(\Gamma)$ and is isotopic to c_i rel endpoints inside $M - N(K_i)$, so that $\gamma_i = c_i \cup d_i$ bounds a disk in $M - N(K_i)$ and has $tb(\gamma_i) = -1$ with respect to this disk. Another way of thinking about the attachment of c_i is to slide both endpoints of c_i towards $F_1 = \partial M$, along $\Gamma_{\partial(M-N(K_i))}$, so that both endpoints of c_i are then placed on $\Gamma_{\partial M}$. Figure 2 depicts the neighborhood of c_i after the isotopy. The disk D with $tb(\partial D) = -1$ is easy to see in this diagram. Once the arcs of the first type are attached, we need to attach arcs $c_i \subset F_{1-\delta}$ of the second type in order to complete the Legendrian 1-skeleton of $F_{1-\delta}$. These c_i connect interior points of arcs of the first type and do not intersect $\Gamma_F \times \{1-\delta\}$. This is given in Figure 3. The top endpoint of c_i in Figure 3 can be moved to the left and the bottom endpoint can be moved to the right, both along $\Gamma_{\partial(M-N(K_i))}$, so that both endpoints now lie on $\Gamma_{\partial M}$. The result is the same situation as given in Figure 2.

Next we consider Step (3), which is depicted in Figure 4. In the figure, the face $F_1 = \partial M$ is in the back, and the thickening of the Legendrian 1-skeleton of $F_{1-\delta}$ is to the front. The thickening of the arc c_i is the cylinder to the left emanating from $\Gamma_{\partial M}$. $\square$

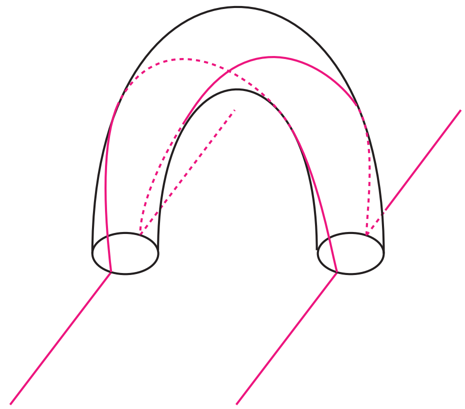

FIGURE 2. The face $F_1 = \partial M$ is in the back.

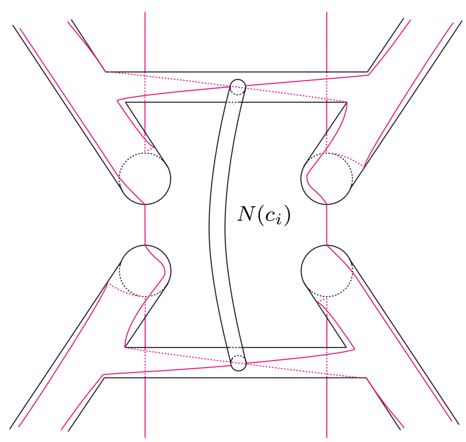

FIGURE 3. Step (1) of the subdivision process. Attaching an arc of the second type. The face $F_1 = \partial M$ is in the back.

2. DEFINITION OF THE CONTACT CLASS

We briefly recall Juhász' sutured Floer homology theory [Ju1, Ju2].

A *compatible* Heegaard splitting for a sutured manifold (M, Γ) consists of a Heegaard surface Σ (not necessarily connected) with nonempty boundary, together with two sets of pairwise disjoint simple closed curves that do not intersect $\partial\Sigma$, the α-curves $\alpha_1, \dots, \alpha_r$ and the β-curves $\beta_1, \dots, \beta_r$. Then M is obtained from $\Sigma \times [-1, 1]$ by gluing compressing disks along $\alpha_i \times \{-1\}$ and along $\beta_i \times \{1\}$, and thickening. We take the suture Γ to be $\partial\Sigma \times \{0\}$.

Let $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_r$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_r$, viewed in $Sym^r(\Sigma)$. Then let $CF(\Sigma, \alpha, \beta)$ be the free $\mathbb{Z}$-module generated by the points $\mathbf{x} = (x_1, \dots, x_r)$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The suture Γ plays the role of the basepoint in sutured Floer homology. Denote by $\mathcal{M}_{\mathbf{x},\mathbf{y}}$ the 0-dimensional (after quotienting by the natural $\mathbb{R}$-action) moduli space of holomorphic maps u from the unit disk $D^2 \subset \mathbb{C}$ to $Sym^r(\Sigma)$ that (i) send $1 \mapsto \mathbf{x}$, $-1 \mapsto \mathbf{y}$, $S^1 \cap \{\mathrm{Im}\ z \geq 0\}$ to $\mathbb{T}_\alpha$ and

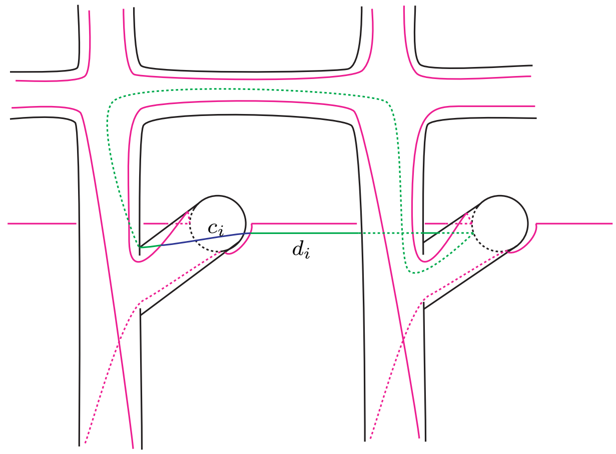

FIGURE 4. Step (3) of the subdivision process. The blue arc is c_i and the green arc is d_i.

$S^1 \cap \{\text{Im } z \leq 0\}$ to $\mathbb{T}_\beta$, and (ii) avoid $\partial\Sigma \times Sym^{r-1}(\Sigma) \subset Sym^r(\Sigma)$. Then define

$$\partial \mathbf{x} = \sum_{\mu(\mathbf{x},\mathbf{y})=1} \#(\mathcal{M}_{\mathbf{x},\mathbf{y}})\ \mathbf{y},$$

where $\mu(\mathbf{x},\mathbf{y})$ is the relative Maslov index of the pair and $\#(\mathcal{M}_{\mathbf{x},\mathbf{y}})$ is a signed count of points in $\mathcal{M}_{\mathbf{x},\mathbf{y}}$. The homology $SFH(M,\Gamma)$ of this complex is shown to be independent of the various choices made in the definition. In particular, it is independent of the choice of a "weakly admissible" Heegaard decomposition.

Interlude on orientation conventions. The convention for (M,Γ) (consistent with that of Ozsváth-Szabó and Juhász) is as follows: The Heegaard surface Σ is an oriented surface whose oriented boundary is Γ. The suture/dividing set Γ is also defined as the boundary of $R_+(\Gamma)$ (= minus the boundary of $R_-(\Gamma)$). If Σ splits M into two compression bodies H_1 and H_2, and $\partial H_1 = \Sigma$, $\partial H_2 = -\Sigma$, then the boundaries of the compressing disks for H_1 are the α_i and the boundaries of the compressing disks for H_2 are the β_i. Then $SFH(M,\Gamma)$ is the Floer homology $SFH(\Sigma,\alpha,\beta)$, in that order. We will now describe $SFH(M,-\Gamma)$, $SFH(-M,\Gamma)$, and $SFH(-M,-\Gamma)$, using the same data (Σ,α,β). The reader is warned that the Σ,α,β that appear in this interlude will be different from the Σ,α,β that appear subsequently.

$SFH(M,-\Gamma)$: If we keep the same orientation for M and switch the orientation of Σ, then we must switch α and β. Hence $SFH(M,-\Gamma) = SFH(-\Sigma,\beta,\alpha)$. Also $R_\pm(\Gamma) = R_\mp(-\Gamma)$.

$SFH(-M,\Gamma)$: If we switch the orientation of M, then $\partial H_1 = -\Sigma$ and $\partial H_2 = \Sigma$. Since the orientation of Γ is unchanged, the orientation of Σ is unchanged. Hence $SFH(-M,\Gamma) = SFH(\Sigma,\beta,\alpha)$. Observe that $R_\pm(M,\Gamma) = R_\mp(-M,\Gamma)$.

$SFH(-M,-\Gamma) = SFH(-\Sigma,\alpha,\beta)$, from the above considerations.

Given (M,Γ,ξ), the decomposition of M into $M - N(K)$ and $N(K)$ from Theorem 1.1 gives us a *partial open book decomposition* $(S, R_+(\Gamma), h)$, which we now describe. (The terminology *partial* open book decomposition is used because M can be constructed from a page S and a partially-defined monodromy map $h : P = S - \overline{R_+(\Gamma)} \to S$.) The tubular

portion T of $-\partial N(K)$ is split by the dividing set into positive and negative regions, with respect to the orientation of $-\partial N(K)$ or $\partial(M - N(K))$. Let P be the positive region. Next, if $D_i \subset \partial N(K)$ are the attaching disks of $N(K)$, then consider $R_+(\Gamma) - \cup_i D_i$; from now on this subsurface of ∂M will be called $R_+(\Gamma)$. Then the page S is obtained from the closure $\overline{R_+(\Gamma)}$ of $R_+(\Gamma)$ by attaching the positive region P. Let $\sim$ be an equivalence relation on $S \times [-1,1]$ given by $(x,t) \sim (x,t')$, where $x \in \partial S$ and $t,t' \in [-1,1]$. Then $S \times [-1,1]/_\sim$ can be identified with $M - N(K)$, where $\Gamma_{\partial(M-N(K))} = \partial S \times \{0\}$. The manifold M is then obtained from $S \times [-1,1]/_\sim$ by attaching thickenings of the compressing disks D_i^β corresponding to the meridians of $N(K)$. The suture Γ on ∂M is $\partial(\overline{R_+(\Gamma)}) \times \{1\}$ — we emphasize that this Γ is no longer quite the same as the dividing set $\Gamma_{\partial M}$. Also observe that $\partial D_i^\beta \cap (\partial(\overline{R_+(\Gamma)}) \times \{1\}) = \emptyset$. The Heegaard surface is $\Sigma = \partial(S \times [-1,1]/_\sim) - (R_+(\Gamma) \times \{1\})$. One set of compressing disks is $\{D_i^\beta\}$ and the other set $\{D_i^\alpha\}$ gives a disk decomposition of $M - N(K)$. Let h be the monodromy map — it is obtained by first pushing P across $N(K)$ to $T - P \subset \partial(M - N(K))$, and then following it with an identification of $M - N(K)$ with $S \times [-1,1]/_\sim$.

Suppose that S is obtained by successively attaching r 1-handles to the union of $\overline{R_+(\Gamma)}$ and the previously attached 1-handles. Let a_i, $i = 1, \dots, r$, be properly embedded arcs in $P = S - \overline{R_+(\Gamma)}$ with endpoints on $A = \partial P - \Gamma$, so that $S - \cup_i a_i$ deformation retracts onto $\overline{R_+(\Gamma)}$. A collection $\{a_1, \dots, a_r\}$ of such arcs is called a *basis* for $(S, R_+(\Gamma))$. In fact, $\{a_1, \dots, a_r\}$ is a basis for $H_1(P, A)$. Next let b_i be an arc which is isotopic to a_i by a small isotopy so that the following hold:

(1) The endpoints of a_i are isotoped along ∂S, in the direction given by the boundary orientation of S.
(2) The arcs a_i and b_i intersect transversely in one point in the interior of S.
(3) If we orient a_i, and b_i is given the induced orientation from the isotopy, then the sign of the intersection $a_i \cap b_i$ is $+1$.

Then the α-curves are $\partial(a_i \times [-1,1])$ and the β-curves are $(b_i \times \{1\}) \cup (h(b_i) \times \{-1\})$, viewed on $S \times [-1,1]/_\sim$. The α-curves and β-curves avoid the suture Γ and hence determine a Heegaard splitting (Σ, β, α), which is easily seen to be weakly admissible. Now the union of Σ and $R_+(\Gamma) = R_+(\Gamma) \times \{1\}$ bound $S \times [0,1]/\sim$. Define the orientation on Σ to be the outward orientation inherited from $S \times [0,1]/\sim$. Then $\partial\Sigma$ is oriented oppositely from $\partial(R_+(\Gamma))$, and we have $\partial\Sigma = -\Gamma$. Hence $SFH(\Sigma, \beta, \alpha) = SFH(-M, -\Gamma)$.

The contact class is basically the *EH* class which was defined in [HKM2]. The only difference is that we are not using a full basis for S and that the contact class sits in $SFH(-M, -\Gamma)/\{\pm 1\}$. Let $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_r$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_r$, viewed in $Sym^r(\Sigma)$. Let $CF(\Sigma, \beta, \alpha)$ be the chain group generated by the points in $\mathbb{T}_\beta \cap \mathbb{T}_\alpha$. Let x_i be the intersection point $(a_i \cap b_i) \times \{1\}$ lying in $S \times \{1\}$. Then $\mathbf{x} = (x_1, \dots, x_r)$ is a cycle in $CF(\Sigma, \beta, \alpha)$ due to the placement of Γ, and its class in $SFH(-M, -\Gamma)$ will be written as $EH(S, h, \{a_1, \dots, a_r\})$. Figures 5 and 6 depict the situation described above.

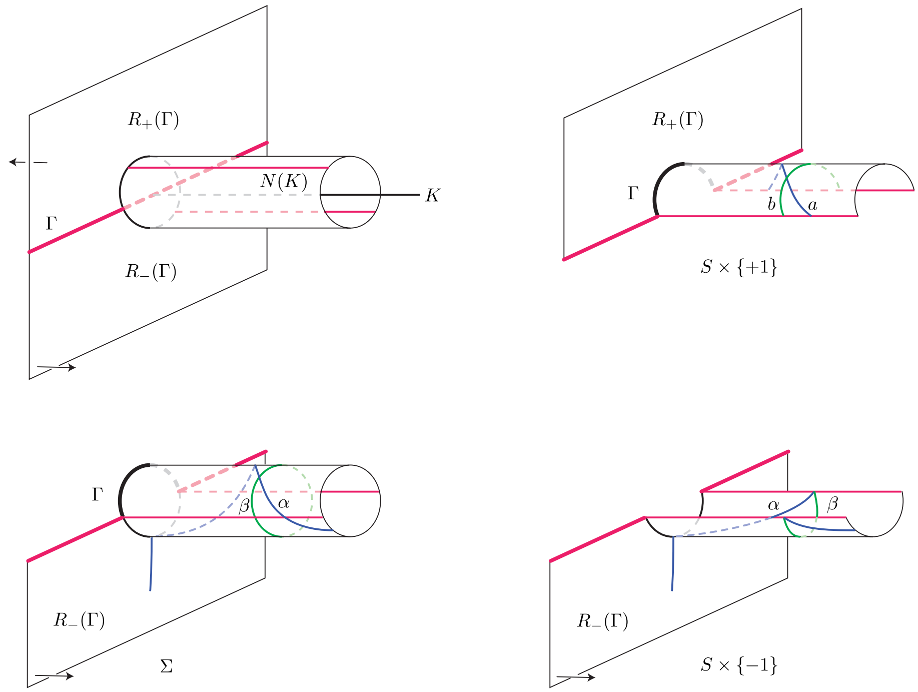

FIGURE 5. The first figure shows a small portion of M near an intersection point of Γ and K and is drawn from the point of view of the interior of $M - N(K)$. The second figure shows the page $S \times \{1\}$ and a basis arc a. Notice that in this figure $R_+(\Gamma)$ denotes a proper subsurface of the previous $R_+(\Gamma)$. The boundary of the new $R_+(\Gamma)$ is the new Γ. We have also applied edge-rounding to connect the dividing curves on $\partial M - \cup_i D_i$ to those on the tubular portion T. The third figure shows the Heegaard surface Σ. Notice that the arc $\beta - b$ is obtained by pushing b through $N(K)$, while $\alpha - a$ is obtained by isotoping a through $M - N(K)$. The subsurface of Σ indicated in the fourth figure is $S \times \{-1\}$. Notice that the single point of intersection $a \cap b$ is replaced by two copies of itself in this figure. In doing sutured Floer homology computations, we only consider holomorphic disks which miss complementary regions of $\alpha \cup \beta$ containing Γ. This ensures that the computations may be done in the subsurface $S \times \{-1\}$ of Σ.

3. Well-definition of the contact class

Theorem 3.1. *$EH(S, h, \{a_1, \dots, a_r\}) \in SFH(-M, -\Gamma)/\{\pm 1\}$ is an invariant of the contact structure.*

Once the invariance is established, the contact class will be written as $EH(M, \Gamma, \xi)$. As usual, there is a ± 1 indeterminacy, which we sometimes suppress.

Outline of Proof. We need to prove that $EH(S, h, \{a_1, \dots, a_r\})$ is (i) independent of the choice of basis $\{a_1, \dots, a_r\}$, (ii) invariant under stabilization, and (iii) only depends on the isotopy class of h. The dependence only on the isotopy class of h is identical to the proof

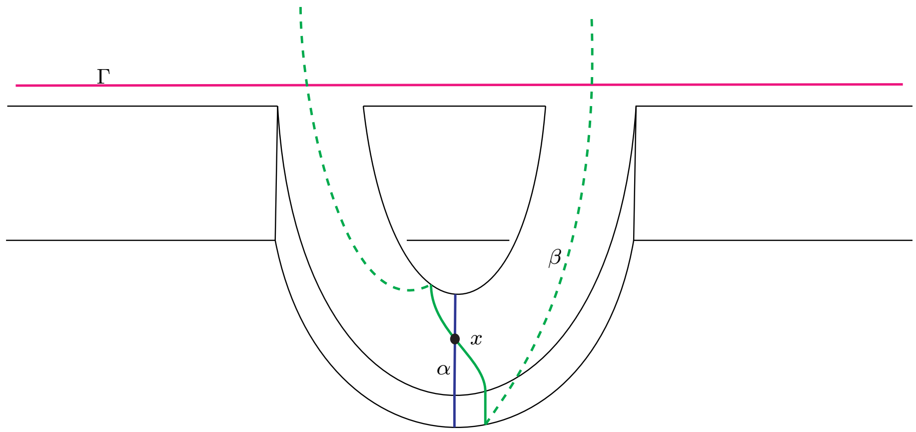

FIGURE 6. The partial open book decomposition. The curve Γ is the suture on ∂M. (It has been pushed into $R_+(\Gamma)$ for better viewing.) The top face is $S \times \{1\}$ and contains $R_+(\Gamma)$. In this figure, a single 1-handle is attached to $\overline{R_+(\Gamma)}$. The point x is then the contact class.

given in [HKM2], and will be omitted. The proof of the independence of choice of basis is similar to that of [HKM2], and we highlight only the differences in Section 3.1.

We define a *positive stabilization* in the relative case as follows: Let c be a properly embedded arc in S; in particular, it can pass through $R_+(\Gamma)$. Attach a 1-handle to S at the endpoints of c to obtain S', and let γ be a closed curve obtained by gluing c and a core of the 1-handle. Then a *positive stabilization* of $(S, R_+(\Gamma), h)$ is $(S', R_+(\Gamma), R_\gamma \circ h)$, where R_γ is a positive Dehn twist about γ. We emphasize that the order of composing R_γ and h is important, since we are not allowed a global conjugation of S when we only have a partial open book.

Every two partial open book decompositions representing the same contact structure ξ become isotopic after performing a sequence of positive stabilizations to each. The proof follows from Theorem 1.2: Let us consider the case when c_0 is attached to K_0. Since $M - N(K) \simeq S \times [-1,1]/_\sim$, Condition 4 of Theorem 1.2 implies that c_0 can be viewed as a Legendrian arc on $S \times \{0\} \subset S \times [-1,1]/_\sim$. Attaching a neighborhood of c_0 to $N(K_0)$ is equivalent to drilling out a standard neighborhood of the Legendrian arc $c_0 \subset S \times \{0\}$. Recall that the monodromy map h sends $P \times \{1\}$ to $S \times \{-1\}$. Since the drilling takes place in the region $S \times [-1,1]$, *after* h is applied, the new monodromy map is $R_\gamma \circ h$.

The invariance under stabilization in the closed case can be argued as follows, once we establish the independence of choice of basis: Suppose the stabilization occurs along the properly embedded arc c in S. Take a basis $\{a_1, \dots, a_r\}$ for S so that all the a_i are disjoint from c. The arc c may be separating or nonseparating (even boundary-parallel), but there is a suitable basis in either case. Then take the basis $\{a_0, a_1, \dots, a_r\}$ for S', where a_0 is the cocore of the 1-handle attached onto S. Then β_0 and α_0 intersect exactly once, and β_0 is disjoint from the other α_i by construction. Although there is a natural chain isomorphism between $CF(\beta, \alpha)$ and $CF(\beta \cup \{\beta_0\}, \alpha \cup \{\alpha_0\})$, where $\alpha = \{\alpha_1, \dots, \alpha_r\}$ and $\beta = \{\beta_1, \dots, \beta_r\}$, it is not clear that the identification is consistent with the stabilization and handleslide maps in [OS1]. In Lemma 3.5 we prove that $EH(S, h, \{a_1, \dots, a_r\})$ is indeed mapped to

$EH(S', h', \{a_0, \dots, a_r\})$ under the stabilization and handleslide maps in [OS1]. (Notice that this proof of invariance could have been used in [HKM2] without appealing to the equivalence with the Ozsváth-Szabó contact invariant. This was pointed out to the authors by András Stipsicz.)

Now, if $R_+(\Gamma)$ is not a homotopically trivial disk, then it is not always possible to find a basis $\{a_1, \dots, a_r\}$ which is disjoint from the arc of stabilization c. This difficulty is dealt with in Section 3.2. □

3.1. **Change of basis.** Let $\{a_1, a_2, \dots, a_r\}$ be a basis for $(S, R_+(\Gamma))$. After possibly reordering the a_i's, suppose a_1 and a_2 are adjacent arcs on $A = \partial P - \Gamma$, i.e., there is an arc $\tau \subset A$ with endpoints on a_1 and a_2 such that τ does not intersect any a_i in $\text{int}(\tau)$. Define $a_1 + a_2$ as the isotopy class of $a_1 \cup \tau \cup a_2$, relative to the endpoints. Then the modification $\{a_1, a_2, \dots, a_r\} \mapsto \{a_1 + a_2, a_2, \dots, a_r\}$ is called an *arc slide.*

Lemma 3.2. *$EH(S, h)$ is invariant under an arc slide $\{a_1, a_2, \dots, a_r\} \mapsto \{a_1 + a_2, a_2, \dots, a_r\}$.*

Proof. Same as that of Lemma 3.4 of [HKM2]. □

Let $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_r\}$ be two bases for $(S, R_+(\Gamma))$. Assume that the two bases intersect transversely and *efficiently*, i.e., each pair of arcs a_i, b_j realizes the minimum number of intersections in its isotopy class, where the endpoints of the arcs are allowed to move in A. In particular, there are no bigons consisting of a subarc of a_i and a subarc of b_j, and no triangles consisting of a subarc of a_i, a subarc of b_j, and a subarc of A.

Lemma 3.3. *Suppose that each component of $\overline{P}$ intersects Γ along at least two arcs. Then there is a sequence of arc slides which takes $\{a_1, \dots, a_r\}$ to $\{b_1, \dots, b_r\}$.*

Here $\overline{P}$ is the closure of P. The condition of the lemma is easily satisfied by performing a trivial stabilization along a boundary-parallel arc. Moreover, a trivial stabilization is easily seen to preserve the EH class.

Proof. Consider a connected component Q of $S - \cup_{i=1}^r a_i - \overline{R_+(\Gamma)}$. Then Q is a (partially open) polygon whose boundary ∂Q consists of $2k$ arcs, $k-1$ of which are a_i or a_i^{-1}, k of which are subarcs $\tau_1, \dots, \tau_k$ of $\partial P - \Gamma$, and one which is a subarc γ of Γ. (For the moment we have oriented the a_i, and the notation a_i^{-1} means that the orientation of a_i and the orientation of ∂Q are opposite.) Since each component of $\overline{P}$ intersects Γ along at least two arcs, there is at least one arc $c \subset \partial Q$ of type a_i or a_i^{-1} so that c^{-1} does not appear on ∂Q. Otherwise, Q glues up to a component of P whose closure intersects Γ along one arc.

Suppose $(\bigcup_{i=1}^r a_i) \cap (\bigcup_{i=1}^r b_i) \neq 0$. We will apply a sequence of arc slides to $\{a_i\}_{i=1}^r$ to obtain $\{a_i'\}_{i=1}^r$ which is disjoint from $\{b_i\}_{i=1}^r$. After possibly reordering the arcs, there is a subarc $b_1^0 \subset b_1$ in Q with endpoints on a_1 and τ_1 on ∂Q. The subarc b_1^0 separates Q into two regions Q_1 and Q_2, only one of which (say Q_1) has γ as a boundary arc. If $c = a_1$, then we can slide c around the boundary of Q_2 to obtain a_1' which has fewer intersections with b_1. If a_1 and a_1^{-1} both occur on ∂Q, then we have a problem if Q_1 has γ as a boundary arc and Q_2 has a_1^{-1} as a boundary arc. To maneuver around this problem, move c via a sequence of arc slides so that the resulting c' is parallel to γ (and protects it). Then we can slide a_1 around the boundary of ∂Q_1 as in [HKM2].

Finally, suppose that $\{a_i\}_{i=1}^r$ and $\{b_i\}_{i=1}^r$ are disjoint. Let Q be a component of $S-\cup_i a_i-\overline{R_+(\Gamma)}$, as before. Let b_1 be an arc in Q which is not parallel to any a_i or a_i^{-1}. The arc b_1 cuts Q into two components Q_1 and Q_2, where Q_1 has γ as a boundary arc. We claim that there is some a_i on ∂Q_2 so that a_i^{-1} is not on ∂Q_2. Otherwise, the arcs on ∂Q_2 will be paired up and b_1 will cut off a subsurface of S whose closure does not intersect Γ. Moreover, we may also assume that a_i is not parallel to any b_j. Now, apply a sequence of arc slides to a_i so that it becomes parallel to b_1. □

3.2. **Stabilization.** A properly embedded arc c in S that intersects Γ efficiently is said to have *complexity* n if there are n subarcs of c in $\overline{P}$, both of whose endpoints are on Γ. Let (S',h') be a positive stabilization of a partial open book decomposition (S,h) along a properly embedded arc $c\subset S$ that intersects Γ efficiently. Then (S',h') is a stabilization of (S,h) of *complexity* n if c has complexity n.

The following lemma gives the fundamental property of arcs of complexity zero:

Lemma 3.4. *An arc c has complexity zero if and only if there is a basis for $(S,R_+(\Gamma))$ which is disjoint from c.*

Proof. If an arc c has complexity >0, then there is a subarc $c_0\subset c$ in $\overline{P}$, both of whose endpoints are on Γ. It is impossible to find a basis that does not intersect c_0, since the pair $(c_0,\partial c_0)$ is homotopically nontrivial in $(\overline{P},\Gamma)$.

On the other hand, suppose the complexity is zero. Then there are at most two subarcs of $c\cap\overline{P}$. If c lies in $\overline{R_+(\Gamma)}$, then it does not intersect any basis of $(S,R_+(\Gamma))$. If c is entirely contained in P, then it is easy to see that there is a basis which avoids c. (There are two cases, depending on whether c cuts off a subsurface of P whose boundary does not intersect Γ.) Let c_i be a subarc of $c\cap\overline{P}$. Depending on c, there may be two such, i.e., $i=1,2$, or only one such, i.e., $i=1$. In either case, for each c_i, one of its endpoints is on Γ and the other on $\partial P-\Gamma$. If c_i is a trivial subarc, i.e., forms a triangle in P, together with an arc of Γ and an arc of $\partial P-\Gamma$, then c_i does not obstruct the formation of a basis and can in fact be isotoped out of P. Therefore assume that c_i is nontrivial. Let τ_1,γ,τ_2 be subarcs of ∂P in counterclockwise order, where $\gamma\subset\Gamma$ contains an endpoint of c_1, and τ_i are subarcs of $\partial P-\Gamma$. Also let τ be the component of $\partial P-\Gamma$ with the other endpoint of c_1. Then take the first arc a of the basis to be parallel to c_1, with endpoints on either τ and τ_1 or τ and τ_2. We have a choice of either of these unless c_2 also has an endpoint on γ, in which case only one of these will work (and not intersect c_2). The nontriviality of c_1 implies that a is neither boundary-parallel, i.e., cuts off a disk in P, nor parallel to Γ. If the arc a cuts off a subsurface of P whose boundary does not intersect Γ, then we must replace it with disjoint, nonseparating arcs $a_1,\dots,a_{2g}$, where g is the genus of the cut-off surface. Once we cut P along a or $a_1,\dots,a_{2g}$, then c_1 becomes trivial in the cut-open surface, and can be ignored. We apply the same technique to c_2, if necessary, and the rest of the basis can be found without difficulty. □

Lemma 3.5. *Let (S',h') be a stabilization of (S,h) along an arc c of complexity zero. Suppose $\{a_1,\dots,a_r\}$ is a basis for S which is disjoint from c and $\{a_0,a_1,\dots,a_r\}$ is a basis for S', where a_0 is the cocore of the 1-handle attached to obtain S'. Then $EH(S,h,\{a_1,\dots,a_r\})$ is*

mapped to $EH(S', h', \{a_0, \dots, a_r\})$ under the stabilization and handleslide maps defined in [OS1].

Proof. If no β_i, $i > 0$, nontrivially intersects α_0, we have a standard stabilization. Suppose there is some β_i, $i > 0$, which nontrivially intersects α_0. This means $h(b_i)$ nontrivially intersected c before stabilization. Let η be a closed curve on S', obtained from c by attaching the core of the 1-handle. In the left-hand diagram of Figure 7, a portion of $h(b_i)$ intersecting c is drawn; $h'(b_i)$ is obtained from $h(b_i)$ by applying R_η. In order to remove its intersection with α_0, we handleslide β_i across β_0 to obtain γ_i, as indicated on the right-hand side of Figure 7. In Figures 7 and 8, we place a dot in a region of $\partial(S \times [-1,1]/\sim) - \cup_{i=0}^r \alpha_i - \cup_{i=0}^r \beta_i$ to indicate that any point in the region has a path in the region that connects to $\Gamma = \partial\Sigma$. In the right-hand diagram of Figure 7, the dot to the left of α_0 is clear; by following β_0, the region with the dot to the right of α_0 is the same as the region with the dot to the left of α_0.

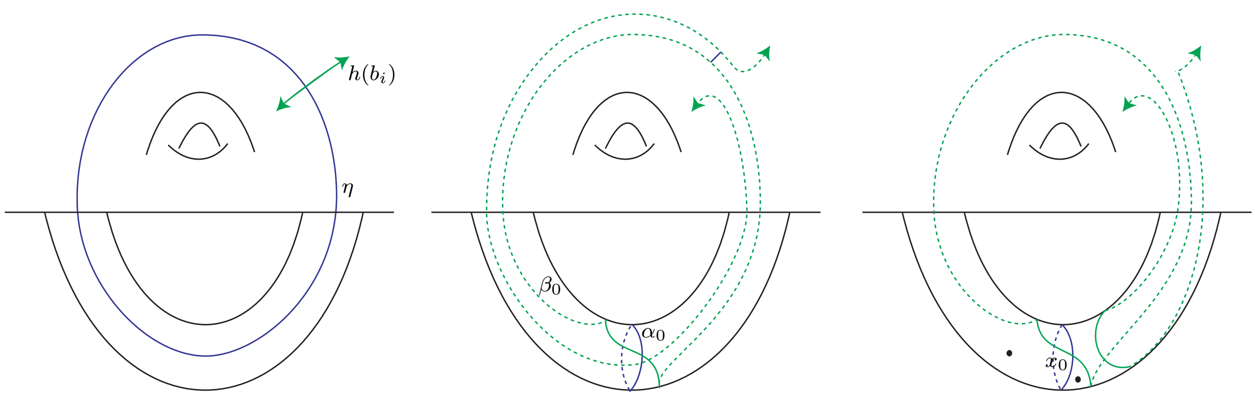

FIGURE 7. The left-hand diagram shows the closed curve η and an intersection point of $h(b_i)$ and η, on a page S. The middle diagram is the Heegaard diagram before the handleslide, and right-hand diagram gives the Heegaard diagram after the handleslide. The dotted curves are on the page $S \times \{-1\}$ and the solid curves are on $S \times \{1\}$. The short blue arc is on $S \times \{-1\}$ and is the handleslide arc.

Let γ_j, $j \neq i$, be small pushoffs of β_j. Figure 8 shows the corresponding arcs in P. The left-hand side of Figure 8 corresponds to $i = 0$ (and Figure 7), and the right-hand side corresponds to $i \neq 0$. We claim that the triple-diagram $(\Sigma, \gamma, \beta, \alpha, \Gamma)$ is weakly admissible. Recall that a triple-diagram is *weakly admissible* if each nontrivial triply-periodic domain which can be written as a sum of doubly-periodic domains has both positive and negative coefficients. Consider the situation on the right-hand side of Figure 8. The connected components of $\Sigma - \cup_i \alpha_i - \cup_i \beta_i - \cup_i \gamma_i$ are numbered as in the diagram. Due to the placement of Γ, the only potential doubly-periodic domain involving β, α in the vicinity of the right-hand diagram is $D_1 + D_2 - D_4 - D_5$. Similarly, the only domain involving γ, β is $D_1 + D_6 - D_3 - D_4$, and the only one involving γ, α is $D_2 + D_3 - D_5 - D_6$. Taking linear combinations, we have

$$\begin{aligned} &a(D_1 + D_2 - D_4 - D_5) + b(D_1 + D_6 - D_3 - D_4) + c(D_2 + D_3 - D_5 - D_6) \\ &= (a+b)D_1 + (a+c)D_2 + (-b+c)D_3 - (a+b)D_4 - (a+c)D_5 + (b-c)D_6. \end{aligned}$$

Since the coefficients come in pairs, e.g., $a+b$ and $-(a+b)$, if any of $a+b$, $b+c$, $a-c$ does not vanish, then the triply-periodic domain has both positive and negative coefficients. Hence, if any of α_i, β_i and γ_i is used for some $i>0$, then we are done. Otherwise, we may assume that none of α_i, β_i and γ_i is used in the periodic domain, for any $i>0$. This allows us to assume that only α_0, β_0 and γ_0 are used for the boundary of the periodic domain. Once all the $\alpha_i, \beta_i, \gamma_i$, $i>0$, are ignored, the D_4 region of the left-hand diagram becomes connected to Γ, and we can apply the same procedure.

We now tensor $\mathbf{x}=(x_0,\dots,x_r)\in CF(\beta,\alpha)$ with the top generator $\Theta\in CF(\gamma,\beta)$. We claim that we obtain $\mathbf{x}'=(x_0',\dots,x_r')$, as depicted in Figure 8. As in the proof of the weak admissibility, we start with the vicinity of right-hand side, i.e., $i>0$. There are no other holomorphic triangles besides the obvious small triangle involving Θ_i, x_i, and x_i', due to the placement of the dots. Once the $\alpha_i, \beta_i, \gamma_i$, $i>0$, are exhausted, D_4 becomes connected to Γ, and we only have the small triangle with vertices Θ_0, x_0, x_0'.

Let the new β be the old γ and let the new $\mathbf{x}$ be the old $\mathbf{x}'$. After a sequence of such handleslides (and renamings), we obtain α and β so that α_0 and β_0 intersect once and do not intersect any other α_i and β_i. The standard stabilization map simply forgets x_0. □

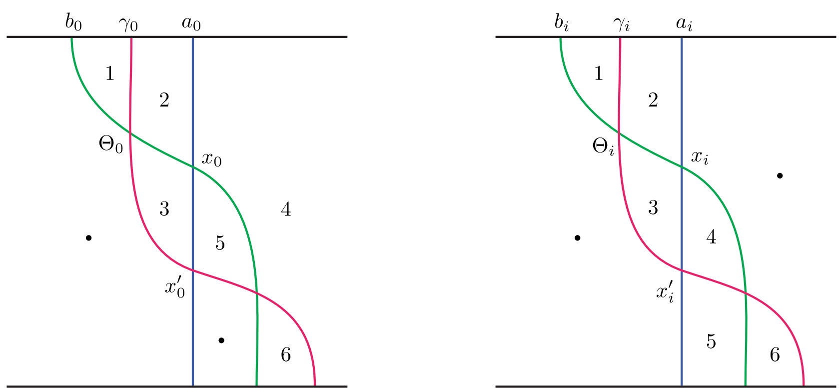

FIGURE 8. The blue arc is a_i, the green arc is b_i, and the red arc is a pushoff of b_i.

Proposition 3.6. *Let (S',h') be a stabilization of (S,h) along an arc c of complexity n. Then there exists a stabilization of complexity at most $n-1$ taking (S',h') to (S'',h''), and two stabilizations of complexity at most $n-1$ which take (S,h) to (S'',h'').*

Proof. A stabilization corresponds to drilling out a neighborhood of a properly embedded Legendrian arc $c=c\times\{0\}\subset S\times\{0\}$ inside $S\times[-1,1]/_\sim$. We may assume that $\Gamma_{\partial(S\times[-1,1]/_\sim)}=\partial S\times\{0\}$ and that $S\times\{0\}$ is a convex surface with dividing set $\Gamma_{S\times\{0\}}=\partial S\times\{0\}$. Then c can be realized as a Legendrian arc on $S\times\{0\}$ after applying the *Legendrian realization principle* (see e.g. [H1]). The nonisolating condition, required in the Legendrian realization principle, is easily met. The arc c may be thought of having twisting number $-\frac{1}{2}$ relative to $S\times\{0\}$, since it begins and ends on the dividing set and intersects no other dividing curves. Our strategy is to attach a Legendrian arc $d\subset P\times\{0\}$ with one endpoint on $(\partial P-\Gamma)\times\{0\}$ and the other endpoint on c to obtain a Legendrian graph $L=c\cup d$ on $S\times\{0\}$. (Again, the nonisolating condition is satisfied.) The endpoint of d on

c must lie on a component of $c \cap \overline{P}$ with both endpoints on Γ. Then $S'' \times [-1,1]/_\sim$ will be $(S \times [-1,1]/_\sim) - N(L)$. We will exploit two ways of obtaining the trivalent graph L. One way is to stabilize along c and then along d. For the other way, subdivide $c = c_1 \cup c_2$ at the trivalent vertex of L. We label the arcs c_1, c_2 so that d, c_1, c_2 are in counterclockwise order about the trivalent vertex. Then first stabilize along $c_1 \cup d$ and then along c_2. See Figure 9.

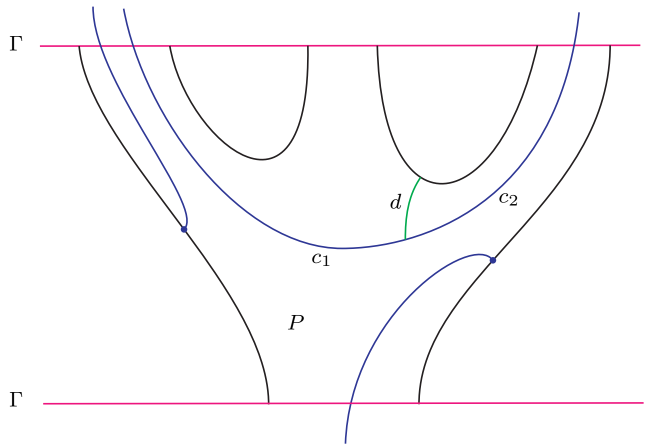

FIGURE 9. The red arcs are Γ and the region drawn is P.

Next refer to Figure 10. We claim that if we stabilize along the arc $c_1 \cup d$ (its thickening is drawn in the diagram to the left), and then stabilize along c_2, given as the green arc, then the two stabilizations are equivalent to the stabilizations on the right-hand side. To see this, we isotop c_2 inside $B = (S \times [-1,1]/_\sim) - N(c_1 \cup d)$, while constraining its endpoints to lie on $\Gamma_{\partial B}$. The key point to observe is that c_2 can be slid "to the right" so that it lies above the neighborhood of $c_1 \cup d$.

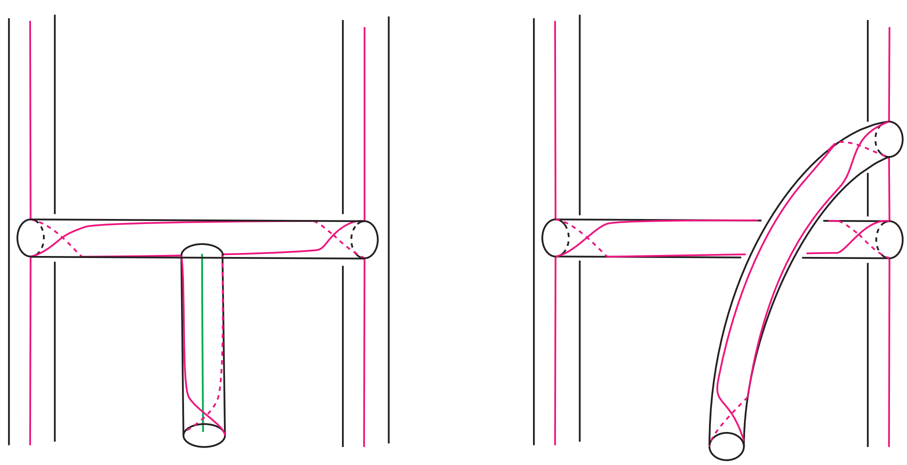

FIGURE 10.

Another way to see this, without the use of contact topology, is the following. Let $c = c_1 \cup c_2$, $c' = c_1 \cup d$, and $c'' = c_2 \cup d$. Also let γ, γ', and γ'' be closed curves as depicted in

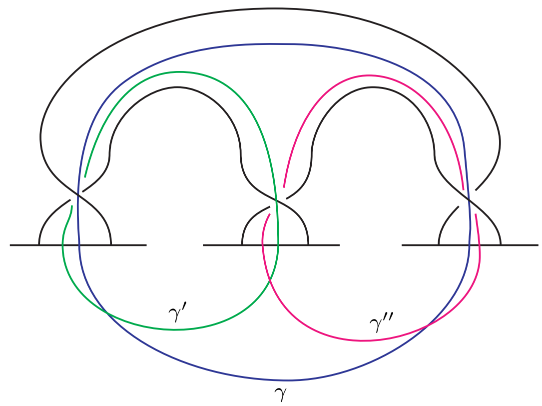

FIGURE 11.

Figure 11, which extend c, c', c''. Then $R_{\gamma'} \circ R_\gamma = R_{\gamma''} \circ R_{\gamma'}$ (on a punctured torus). (Here R_γ is the positive Dehn twist about γ.) □

Proposition 3.7. *Let (S', h') be a stabilization of (S, h) along an arc c of complexity n. Then $EH(S, h)$ is mapped to $EH(S', h')$ under the stabilization and handleslide maps defined in* [OS1].

Proof. Let (Σ, β, α) be a compatible Heegaard splitting for (M, Γ) corresponding to some basis for (S, h). (Similarly define $(\Sigma, \beta^{(i)}, \alpha^{(i)})$, for $(S^{(i)}, h^{(i)})$, defined subsequently.) Suppose (S', h') is a stabilization of (S, h) of complexity zero. Then the map $\Phi : SFH(\beta, \alpha)/\{\pm 1\} \to SFH(\beta', \alpha')/\{\pm 1\}$, defined as a composition of handleslide maps and stabilization maps, sends $EH(S, h) \mapsto EH(S', h')$ by Lemma 3.5 and the invariance under basis change. Assume inductively that stabilizations of complexity at most $n - 1$ take EH classes to EH classes. Then, by Proposition 3.6 (we are using the same notation as the proposition), the map $\Phi_{12} : SFH(\beta', \alpha')/\{\pm 1\} \to SFH(\beta'', \alpha'')/\{\pm 1\}$ sends $EH(S', h') \mapsto EH(S'', h'')$ and the map $\Phi_{02} : SFH(\beta, \alpha)/\{\pm 1\} \to SFH(\beta'', \alpha'')/\{\pm 1\}$ sends $EH(S, h) \mapsto EH(S'', h'')$. Now, $\Phi_{01} : SFH(\beta, \alpha)/\{\pm 1\} \to SFH(\beta', \alpha')/\{\pm 1\}$ satisfies $\Phi_{02} = \Phi_{12} \circ \Phi_{01}$ by Theorem 2.1 of [OS5], which states that the maps Φ_{ij} do not depend on the particular sequence of handleslides, stabilizations, and isotopies chosen. This implies that Φ_{01} maps $EH(S, h) \mapsto EH(S', h')$. □

4. PROPERTIES OF THE CONTACT CLASS

In this section we collect some basic properties of the contact class $EH(M, \Gamma, \xi)$. Most of the properties are analogs of properties of the contact class that are well-known in the case when M is closed. The theorem which does not have an analog in the closed case (for obvious reasons) is the restriction theorem (Theorem 4.5).

Consider a partial open book decomposition (S, h) for (M, Γ, ξ). The notion of a *right-veering* (S, h) can be defined in the same way as in [HKM1]: If for every $x \in \partial P - \Gamma$ and every properly embedded arc $a \subset P$ which begins at x and has both endpoints on $\partial P - \Gamma$, $h(a)$ is to the right of a, then we say (S, h) is *right-veering.*

Proposition 4.1. *If (S, h) is not right-veering, then (M, Γ, ξ) is overtwisted. Any overtwisted (M, Γ, ξ) admits a partial open book decomposition (S, h) which is not right-veering.*

Proof. The first assertion is proved in the same way as the analogous statement in [HKM1].

For the second assertion, note that Example 1 of Section 5 gives a partial open book decomposition of a neighborhood of an overtwisted disk with a left-veering arc. Attach the neighborhood of a Legendrian arc a which connects the boundary of the overtwisted disk to $\Gamma \subset \partial M$, and then complete it to a partial open book for (M, Γ). The left-veering arc from Example 1 survives to give a left-veering arc. □

Proposition 4.2. *If (M, Γ, ξ) admits a partial open book decomposition (S, h) which is not right-veering, then $EH(M, \Gamma, \xi) = 0$.*

Proof. Same as that of [HKM2]. □

By combining Propositions 4.1 and 4.2, we obtain the following:

Corollary 4.3. *If (M, Γ, ξ) is overtwisted, then $EH(M, \Gamma, \xi) = 0$.*

The next proposition describes the effect of Legendrian surgery on the contact invariant.

Proposition 4.4. *If $EH(M, \Gamma, \xi) \neq 0$ and (M', Γ', ξ') is obtained from (M, Γ, ξ) by a Legendrian (-1)-surgery along a closed Legendrian curve L, then $EH(M', \Gamma', \xi') \neq 0$.*

Proof. We can easily extend L to a Legendrian skeleton K as given in Theorem 1.1. Hence there is a partial open book decomposition $(S, R_+(\Gamma), h)$ so that $L \subset P$. Take a basis $\{a_1, \dots, a_r\}$ so that a_1 intersects L once and $a_i \cap L = \emptyset$ for $i > 1$. Push the a_i off to obtain b_i and c_i, as drawn in Figure 12. Let $\alpha_i = (a_i \times \{1\}) \cup (a_i \times \{-1\})$, $\beta_i = (b_i \times \{1\}) \cup (h(b_i) \times \{-1\})$, and $\gamma_i = (c_i \times \{1\}) \cup (R_\gamma \circ h(c_i) \times \{-1\})$. Here γ is the curve $h(L)$ on the page S.

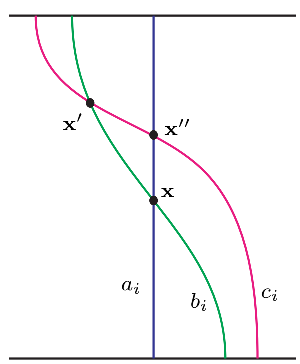

FIGURE 12.

Let $\mathbf{x} \in \mathbb{T}_\beta \cap \mathbb{T}_\alpha$, $\mathbf{x}' \in \mathbb{T}_\gamma \cap \mathbb{T}_\beta$ and $\mathbf{x}'' \in \mathbb{T}_\gamma \cap \mathbb{T}_\alpha$ be the unique r-tuples which are on $S \times \{1\}$. As Baldwin observed in [Ba], there is a comultiplication $CF(\gamma, \alpha) \to CF(\gamma, \beta) \otimes CF(\beta, \alpha)$ which takes $\mathbf{x}''$ to $\mathbf{x}' \otimes \mathbf{x}$. We are assuming that $[\mathbf{x}] \neq 0$. Also, $[\mathbf{x}'] \neq 0$ is immediate from the fact that β_1 and γ_1 intersect only at x_1', due to our choice of basis. Therefore, we have $[\mathbf{x}''] \neq 0$. □

In [H2], the first author exhibited a tight contact structure on a handlebody which became overtwisted after Legendrian surgery. By Proposition 4.4, its contact invariant must vanish.

Theorem 4.5. *Let (M,ξ) be a closed contact 3-manifold and $N \subset M$ be a compact submanifold (without any closed components) with convex boundary and dividing set Γ. If $EH(M,\xi) \neq 0$, then $EH(N,\Gamma,\xi|_N) \neq 0$.*

Proof. Consider the partial open book decomposition $(S, R_+(\Gamma), h)$ for $(N,\Gamma,\xi|_N)$, obtained by decomposing N into $N(K)$ and $N - N(K)$. Now, the complement $N' = M - N$ can be similarly decomposed into $N(K')$ and $N' - N(K')$, where K' is a Legendrian graph in N' with univalent vertices on Γ. We may assume that the univalent vertices of K' and K do not intersect. Then an open book decomposition for (M,ξ) can be obtained from the Heegaard decomposition $(N - N(K)) \cup N(K')$ and $(N' - N(K')) \cup N(K)$. Indeed, both handlebodies are easily seen to be product disk decomposable. The page T for (M,ξ) can be obtained from $(S, R_+(\Gamma), h)$ by successively attaching 1-handles, subject to the condition that none of the handles be attached along ∂P, where $P = S - \overline{R_+(\Gamma)}$. The monodromy map $g : T \to T$ extends $h : P \to S$.

Let $\{a_1, \dots, a_r, a'_1, \dots, a'_s\}$ be a basis for T which extends a basis $\{a_1, \dots, a_r\}$ for $(S, R_+(\Gamma))$. For such an extension to exist, $T - P$ must be connected. This is possible if $M - N$ is connected — simply take suitable stabilizations to connect up the components of $T - P$. If $M - N$ is disconnected, we apply a standard contact connected sum inside $M - N$ to connect up disjoint components of $M - N$. This has the effect of attaching 1-handles to T away from P and extending the monodromy map by the identity. The contact manifold (M,ξ) has been modified, but it is easy to see that the contact class of the connected sum is nonzero if and only if the original contact class is nonzero.

Let $\mathbf{x}$ be the generator of $EH(N,\Gamma,\xi|_N)$ with respect to $\{a_1, \dots, a_r\}$ and $(\mathbf{x}, \mathbf{x}')$ be the generator of $EH(M,\xi)$ with respect to $\{a_1, \dots, a_r, a'_1, \dots, a'_s\}$. If $\partial(\sum_i c_i \mathbf{y}_i) = \mathbf{x}$, then we claim that $\partial(\sum_i c_i(\mathbf{y}_i, \mathbf{x}')) = (\mathbf{x}, \mathbf{x}')$. Indeed, each of the intersection points of $\mathbf{x}'$ must map to itself via the constant map — this uses up all the intersection points of $\mathbf{x}'$. We then erase all the α_i and β_i corresponding to $\mathbf{x}'$, and are left with the Heegaard diagram for $(S, R_+(\Gamma))$. □

Comparison with other invariants. We now make some remarks on the relationship to the contact invariant in the closed case and to Legendrian knot invariants.

1. If we start with a closed (M,ξ), then we can remove a standard contact 3-ball B^3 with $\Gamma_{\partial B^3} = S^1$. The sutured manifold is called $M(1)$ in Juhász [Ju1]. In this case, $SFH(-M(1)) = \widehat{HF}(-M)$, and the contact element in $SFH(-M(1))$ coincides with the Ozsváth-Szabó contact class in $\widehat{HF}(-M)$. (Think of the disk R_+ being squashed to a point to give the basepoint z.)

2. A Legendrian knot L has a standard neighborhood $N(L)$ which has convex boundary. The dividing set $\Gamma_{\partial N(L)}$ satisfies $\#\Gamma_{\partial N(L)} = 2$ and $|\Gamma_{\partial N(L)} \cap \partial D^2| = 2$, where D^2 is the meridian of $N(L)$. Moreover, the framing for L induced by ξ agrees with the framing induced by the ribbon in $N(L)$ which contains L and has boundary on $\Gamma_{\partial N(L)}$.

If (M,ξ) is closed, then $EH(M - N(L), \Gamma_{\partial N(L)}, \xi)$ is an invariant of the Legendrian knot L which sits in $SFH(-(M - N(L)), -\Gamma_{\partial N(L)})$. On the other hand, if we choose the suture Γ on $\partial N(L)$ to consist of two parallel meridian curves, then $SFH(-(M - N(L)), -\Gamma) =$

$\widehat{HFK}(-M,L)$. Hence we are slightly off from the Legendrian knot invariants of Ozsváth-Szabó-Thurston [OST] and Lisca-Ozsváth-Stipsicz-Szabó [LOSS] — the Legendrian knot invariant currently does not sit in $\widehat{HFK}(-M,L)$. We also only have one invariant, rather than two.

5. Examples

In this section we calculate a few basic examples.

Example 1: Standard neighborhood of an overtwisted disk. Let D be a convex surface which is a slight outward extension of an overtwisted disk. Then consider its $[0,1]$-invariant contact neighborhood $M=D\times[0,1]$. After rounding the edges, we obtain $\Gamma_{\partial M}$ which consists of three closed curves, two which are $\gamma\times\{0,1\}$, where γ is a closed curve in the interior of D^2, and one which is $\partial D^2\times\{\frac{1}{2}\}$. Our Legendrian graph K is an arc $\{p\}\times[0,1]$, where $p\in\gamma$. Then $M-N(K)$, after edge-rounding, is a solid torus whose dividing set consists of two parallel homotopically nontrivial curves, each of which intersects the meridian once. See the left-hand diagram of Figure 13 for $M-N(K)$. The meridian of $M-N(K)$ gives a product disk decomposition, and the decomposition into $M-N(K)$ and $N(K)$ gives rise to a partial open book decomposition. The left-hand diagram of Figure 13 shows the arc a on P. The arc a is isotoped through the fibration $N(K)$ (rel endpoints) and then through the fibration $M-N(K)$ (rel endpoints). The resulting arc on $S\times\{1\}$ is $h(a)$.

Next consider the right-hand diagram of Figure 13, which shows a page S of the partial open book decomposition. The region $R_+(\Gamma)\subset S$ has two connected components — an annulus and a disk — the red curves denote their boundary Γ. The region P is the 1-handle which connects the annulus and the disk. The blue arc is the arc $a\times\{-1\}$ and the green arc represents $h(b)\times\{-1\}$, both viewed as sitting on $S\times\{-1\}$. The unique intersection point x between $a\times\{1\}$ and $b\times\{1\}$ is now viewed as two points on $S\times\{-1\}$, both labeled x. (This way, all of the holomorphic disk counting can be done on $S\times\{-1\}$.)

Now, $CF(\beta,\alpha)$ is generated by two points x,y. (This is because $h(a)$ is to the left of a at one of its endpoints.) It is easy to see that $\partial y=x$, so $SFH(-M,-\Gamma)=0$ and $EH(M,\Gamma,\xi)=0$.

Example 2: A $[0,1]$-invariant neighborhood M of a convex surface F, none of whose components $F-\Gamma_F$ are disks. Figure 14 depicts the case when F is a torus with two parallel dividing curves. Let γ_1 and γ_2 be the two components of Γ_F. Then take $K=\{p_1,p_2\}\times[0,1]$, where $p_i\in\gamma_i$, $i=1,2$. It is not hard to see that $M-N(K)$ is product disk decomposable. The page S is obtained from $R_+(\Gamma)$, which is a disjoint union of two annuli, by attaching two bands as in Figure 14. Hence P consists of two connected components, one with a cocore a_1 and the other with a cocore a_2. The black arcs on the left-hand diagram of Figure 14 are obtained from a_i by isotopy through $N(K)$ rel endpoints, and the green arcs $h(a_i)$ by further isotoping through $M-N(K)$. The arc $h(a_i)$ is obtained from a_i via what looks like a "half positive Dehn twist" about a boundary component. The arc $h(a_1)$ only intersects a_1 and the arc $h(a_2)$ only intersects a_2; hence $h(a_1),a_1$ and $h(a_2),a_2$ are independent. To determine $SFH(-M,-\Gamma)$ and verify that $EH(M,\Gamma,\xi)\neq 0$, the only nontrivial regions that need to be considered are D_1 and D_2, which are annuli with corners. (All other domains nontrivially

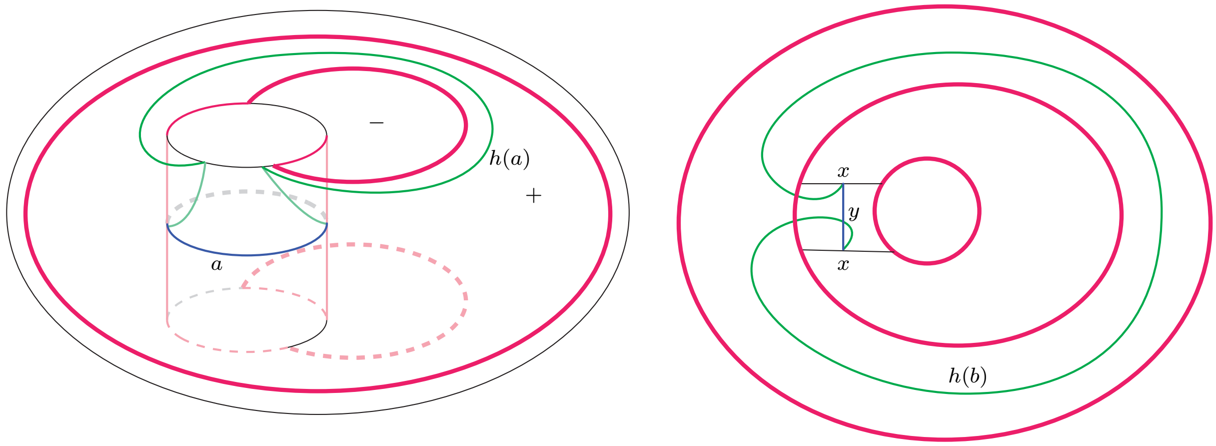

FIGURE 13.

intersect the suture Γ.) We now apply Lipshitz' formula (Corollary 4.10 of [Li1])

$$\mu(D_i) = n_{\mathbf{x}}(D_i) + n_{\mathbf{y}}(D_i) + e(D_i),$$

for computing the index of D_i. Here $e(D_i)$ is the *Euler measure* of D_i and $n_{\mathbf{x}}(D_i)$ is the (weighted) intersection number of the r-tuple $\mathbf{x}$ with D_i. We compute that $\mu(D_i) = 2(\frac{1}{4}) + 2(\frac{1}{4}) - 1 = 0$, which has index $\neq 1$. Hence it follows that $SFH(-M, -\Gamma) = \mathbb{Z}^4 = \mathbb{Z}^2 \otimes \mathbb{Z}^2$ and $EH(M, \Gamma, \xi) \neq 0$. (The fact that we can view $SFH(-M, -\Gamma)$ as $\mathbb{Z}^2 \otimes \mathbb{Z}^2$ follows from the observation that the two intersection points of α_1 and β and the two intersection points of α_2 and β are independent. We can also observe that (M, Γ) admits an annulus decomposition into two solid tori of the type considered in the next example with $n = 2$, and each sutured solid torus contributes $\mathbb{Z}^2$. Then we can use the tensor product formula, proved by Juhász [Ju2, Proposition 8.10].) The general situation is similar.

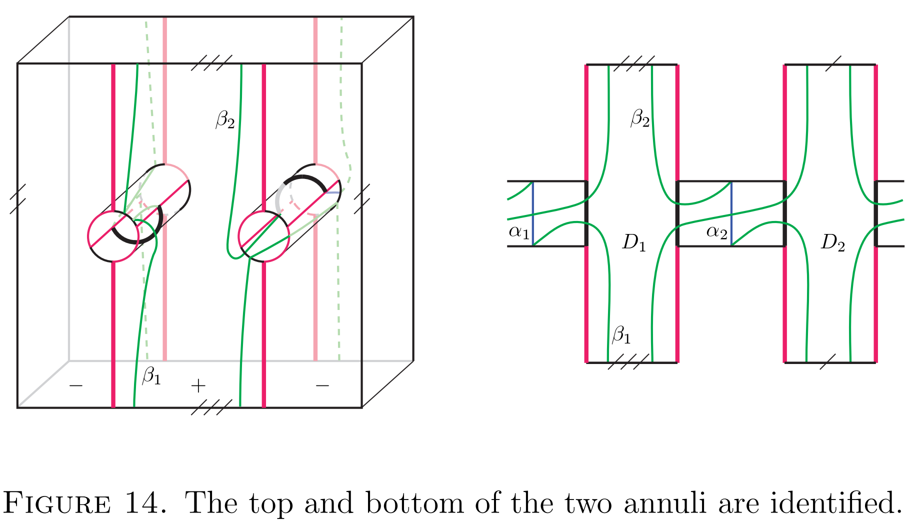

FIGURE 14. The top and bottom of the two annuli are identified.

Example 3: Solid torus $M = S^1 \times D^2$ with slope ∞ and $\#\Gamma = 2n > 2$. According to the classification of tight contact structures on the solid torus [Gi3, H1], there is a 1-1

correspondence between isotopy classes of tight contact structures on (M, Γ) rel boundary and isotopy classes of dividing sets on the meridian disk $D^2 = \{pt\} \times D^2$. Figure 15 depicts the case $n = 4$. There are two tight contact structures with ∂-parallel Γ_{D^2}. (A convex surface T has *∂-parallel* dividing set Γ_T if each component of Γ_T cuts off a half-disk which intersects no other component of Γ_T.) The contact structures with ∂-parallel set are distinguished by their relative Euler class. The monodromy map h in the ∂-parallel case is calculated in Figure 15. On the right-hand side of Figure 15, a_1, a_2, a_3 is a basis for $(S, R_+(\Gamma), h)$, in counterclockwise order. (The α_i are $\partial(a_i \times [-1, 1])$.) Label the intersections $\alpha_1 \cap \beta$ by $x_1, x_2, x_3 = x_1$, $\alpha_2 \cap \beta$ by $y_1, \dots, y_5 = y_1$, and $\alpha_3 \cap \beta$ by $z_1, \dots, z_5 = z_1$, all in *clockwise* order. Starting with $\alpha_1 \cap \beta$, we find that the only valid 3-tuples are (x_i, y_j, z_k), where $i, j, k = 1, 2$. There are regions of $\Sigma - \cup_i \alpha_i - \cup_i \beta_j$ which do not intersect Γ — all such regions are quadrilaterals, but use the same α_i or β_i twice, so are not valid. Hence there are no holomorphic disks, and the boundary map is the zero map. Therefore, $SFH(-M, -\Gamma) = \mathbb{Z}^8 = \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ and $EH(M, \Gamma, \xi) \neq 0$. Moreover, if we split $SFH(-M, -\Gamma)$ according to relative Spin^c-structures, then we have a direct sum $\mathbb{Z} \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}$. The two ∂-parallel tight contact structures use up the first and last $\mathbb{Z}$ summands, and the others have EH classes that live in the remaining $\mathbb{Z}^3 \oplus \mathbb{Z}^3$. We leave it as an exercise to verify that the EH classes of the remaining tight contact structures are nonzero. There are $4 + 2$ tight contact structures ξ_i, $i = 1, \dots, 6$, corresponding to each $\mathbb{Z}^3$ summand, but we expect the EH classes to distinguish them — this means that we believe the $EH(\xi_i)$ are linearly dependent.

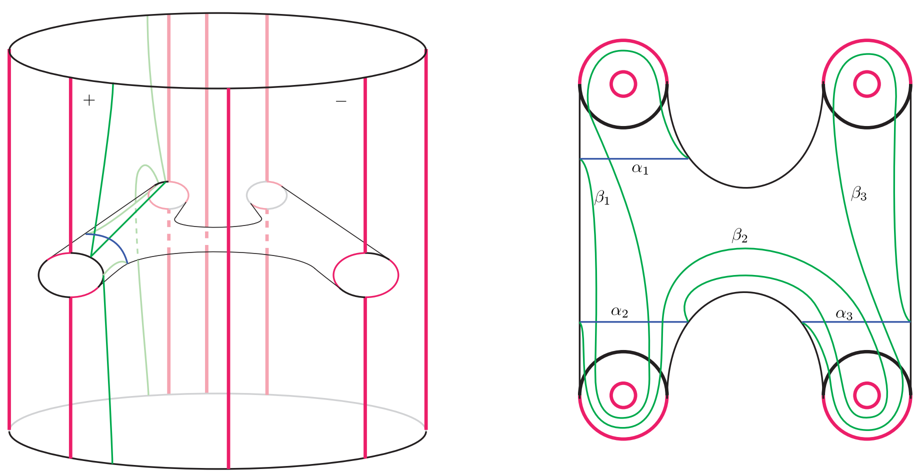

FIGURE 15.

Example 4: A basic slice $M = T^2 \times [0, 1]$ (see [H1]). Write $T_i = T^2 \times \{i\}$, $i = 0, 1$, and take an oriented identification $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$. Normalize so that $\#\Gamma_{T_i} = 2$, Γ_{T_i} are linear, and slope$(\Gamma_{T_1}) = 0$, slope$(\Gamma_{T_0}) = \infty$. A basic slice can be obtained from the convex torus T_0 by attaching a single bypass along a linear arc of slope $-1 < s < 0$ and thickening. Pick a point p on the connected component of Γ_{T_0} which contains the endpoints of the bypass arc of attachment, as indicated in Figure 16. Then we let K be p times an interval. It is not hard to see that $M - N(K)$ is product disk decomposable.

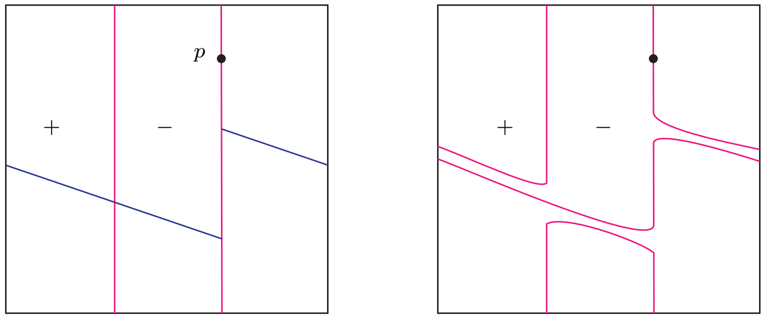

FIGURE 16. The left-hand diagram gives T_0 and the arc of attachment for the bypass, which is attached from the front. The right-hand diagram gives T_1. Here the sides are identified and the top and the bottom are identified to give T^2. The other basic slice is obtained by switching the $+$ and $-$ signs.

The page S is a thrice-punctured sphere obtained from $R_+(\Gamma)$, a disjoint union of two annuli, by adding a 1-handle to connect the two annuli. If $a \subset P$ is the cocore of the 1-handle, then we compute that h is a positive Dehn twist about the connected component of ∂S which contains the endpoints of a. See Figure 17. One easily computes that $SFH(-M, -\Gamma) = \mathbb{Z}^4$ and $EH(M, \Gamma, \xi) \neq 0$. There are two basic slices, which are distinguished by the relative Euler class. They account for $\mathbb{Z} \oplus \mathbb{Z} \subset SFH(-M, -\Gamma)$. The remaining $\mathbb{Z} \oplus \mathbb{Z}$ come from tight contact structures with an extra π-rotation, and are also distinguished by their relative Euler classes. (See Example 6.)

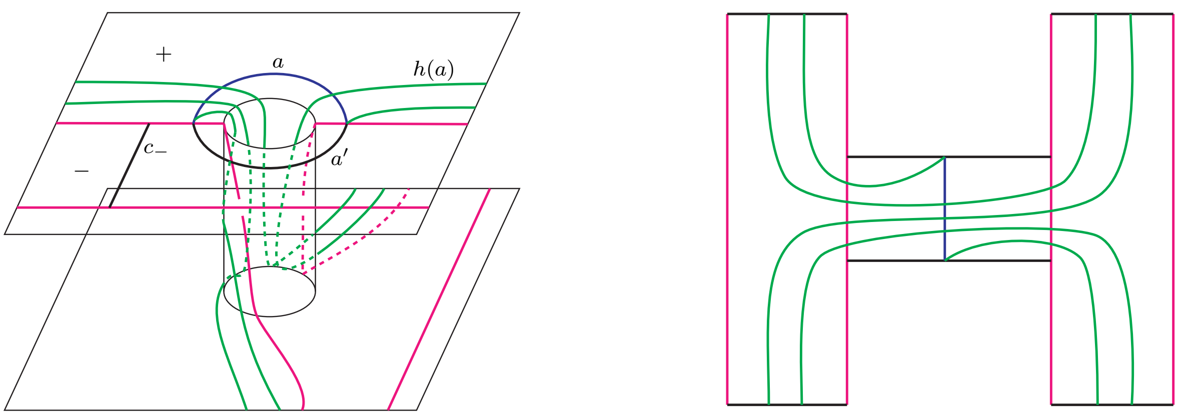

FIGURE 17. The left-hand diagram calculates the monodromy map h. The arc a is pushed across $N(K)$ to give a'. Then a' is pushed across $M - N(K)$ to give $h(a)$. The right-hand diagram gives the monodromy map for a basic slice. Here the top and the bottom of the two annuli are identified.

Example 5: Attaching a bypass. Consider the contact structure (M, Γ, ξ). Let D be a bypass which is attached to M from the exterior, along a Legendrian arc of attachment $c \subset \partial M$. If $M' = M \cup N(D)$ and (M', Γ', ξ') is the resulting contact 3-manifold with convex boundary, then we express the monodromy map $(S', R_+(\Gamma'), h')$ for (M', Γ', ξ') in terms of

$(S, R_+(\Gamma), h)$ for (M, Γ, ξ). Assume without loss of generality that c does not intersect the neighborhood $N(K)$ of the Legendrian skeleton K for M. Let p_1, p_2 be the endpoints of c on Γ and let $c_\pm = c \cap \overline{R_\pm(\Gamma)}$. Next let $\partial D = c \cup d$, where c and d intersect only at their endpoints p_1, p_2. The key observation is that a bypass attachment corresponds to two handle attachments: a 1-handle $N(d)$ attached at p_1 and p_2, followed by a canceling 2-handle. Then $(M - N(K)) \cup N(d)$ is product disk decomposable and gives a fibration structure $S' \times [-1, 1]$. Let a be the arc in P' given in Figure 18. Then, after isotoping it through the 2-handle, a is isotopic rel endpoints to $a' \cup a''$, where a' lies in the positive region and a'' lies in the negative region of $\partial((M - N(K)) \cup N(d))$. Next, isotop a'' through the fibration $M - N(K)$ to obtain a''' which lies in the positive region of $\partial(M - N(K))$. Then $h'(a) = a' \cup a'''$. All the other arcs of P', i.e., those that were in P, are unaffected. Summarizing, S' is obtained from S by attaching a 1-handle from p_1 to p_2 as in Figure 19, the new P' is the union of P and a neighborhood of the arc a, $h'(a) = a' \cup a'''$, and $h' = h$ on P.

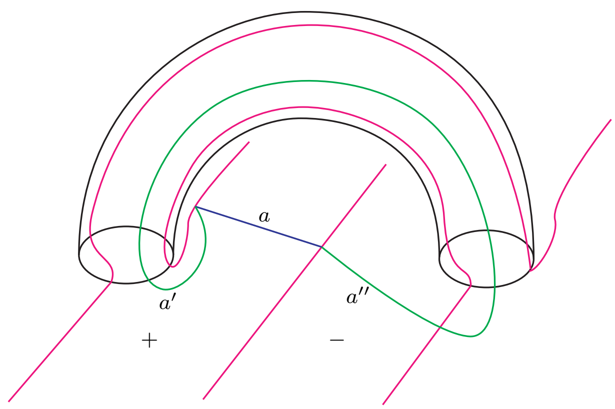

FIGURE 18.

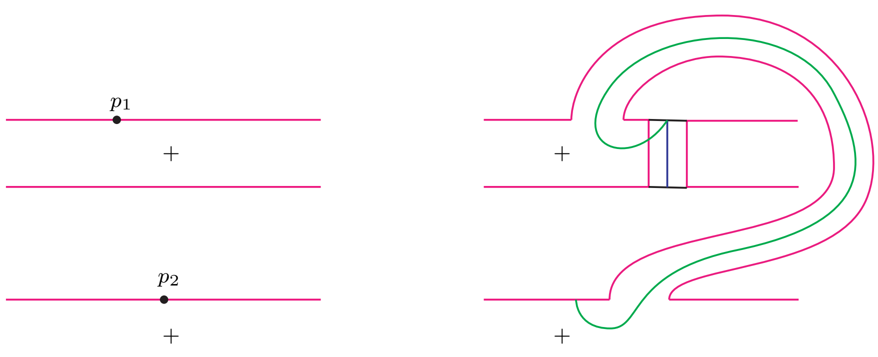

FIGURE 19. The left-hand diagram shows a portion of S before bypass attachment and the right-hand diagram is S', after modification. The green arc is a' and the blue arc is a.

Example 6: Tight contact structures on $T^2 \times I$. Using Examples 4 and 5, we analyze partial open book decompositions obtained from a basic slice by successively attaching bypasses.

These calculations were motivated by computations of open books for tight contact structures on T^3 by Van Horn-Morris [VH], and echo the results obtained there. In particular, we will build up to an open book for the following tight contact structure on $T^2 \times [0,4]$: If (x,y,t) are coordinates on $T^2 \times [0,4]$, then consider $\ker\alpha$, where $\alpha = \cos(\frac{\pi}{2}t)dx - \sin(\frac{\pi}{2}t)dy$. Perturb $\ker\alpha$ along $\partial(T^2\times[0,4])$ so that so it becomes convex with $\#\Gamma_{T_0} = \#\Gamma_{T_4} = 2$, and $\mathrm{slope}(\Gamma_{T_0}) = \mathrm{slope}(\Gamma_{T_4}) = \infty$.

(a) Start with the basic slice $M = T^2 \times [0,1]$ from Example 4, depicted in Figure 16. Take a linear Legendrian arc c of slope $1 < s < \infty$ so that the component of Γ_{T_1} that contains the endpoints of c is different from that containing p. (If the endpoints of c are on the same component as p, then the resulting contact structure is overtwisted.) See the left-hand diagram of Figure 21. Attaching a bypass along c and thickening yields $M' = T^2 \times [0,2]$ with $\#\Gamma_{T_i} = 2$, $\mathrm{slope}(\Gamma_{T_i}) = 0$, $i = 0, 2$. Moreover, the contact structure is obtained from the restriction of α to $T^2 \times [0,2]$, by perturbing the boundary. Hence it has "torsion" $\approx \pi$, where we indicate that the term is used informally by the use of quotations. If $c = c_+ \cup c_-$, where $c_\pm = c \cap \overline{R_\pm(\Gamma_{T_1})}$, then we obtain a (as in Example 5) by pushing c_+ slightly to the right. We isotop c_- to c'_+ in the positive region of $\partial(M - N(p))$ — this is readily done by referring to Figure 17. The result is $h'(a) = c_+c'_+$ as shown in Figure 20. The monodromy

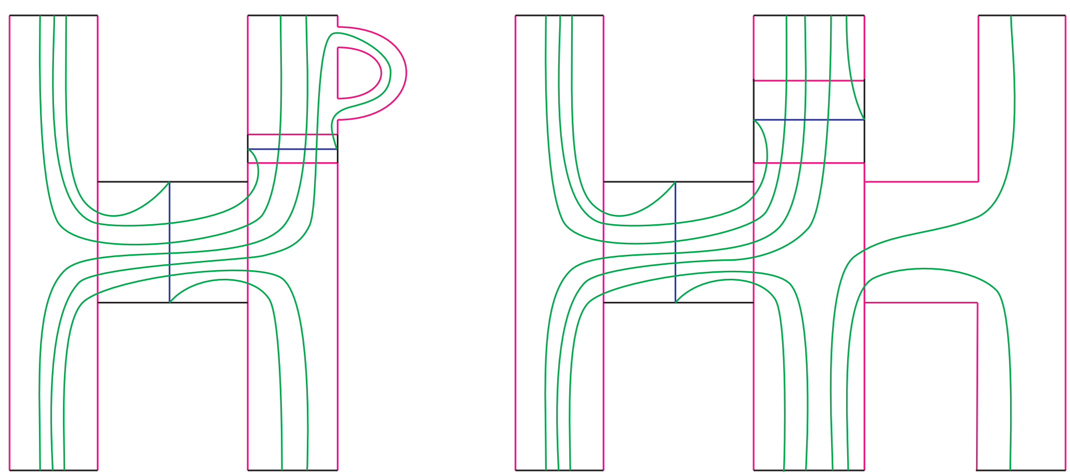

FIGURE 20. The monodromy map. The right-hand diagram is the same as the left-hand one, but drawn more symmetrically. The top and bottom of each annulus is identified.

h' is the restriction of $R_{\gamma_1}R_{\gamma_2}R_{\delta_1}^{-2}$ to P, where γ_1 and γ_2 are parallel to the two boundary components in the middle and δ_1 is the core curve of the middle vertical annulus. (Refer to the right-hand diagram.) Since the only regions of $\Sigma - \cup_i\alpha_i - \cup_i\beta_i$ which do not intersect the suture Γ' for M' are quadrilaterals, it is easy to compute that $EH \neq 0$. The contact class lives in $SFH(-M', -\Gamma') = \mathbb{Z}^2 \otimes \mathbb{Z}^2$ from Example 2 (but is in a different $\mathbb{Z}$ summand).

(b) Next attach another bypass along T_2 to obtain $M'' = T^2 \times [0,3]$, where $\#\Gamma_{T_0} = \#\Gamma_{T_3} = 2$, $\mathrm{slope}(\Gamma_{T_0}) = 0$, and $\mathrm{slope}(\Gamma_{T_3}) = \infty$. The contact structure has "torsion" $\approx \frac{3\pi}{2}$. See Figure 21. Doing a similar calculation as before, we obtain Figure 22. The monodromy h'' is

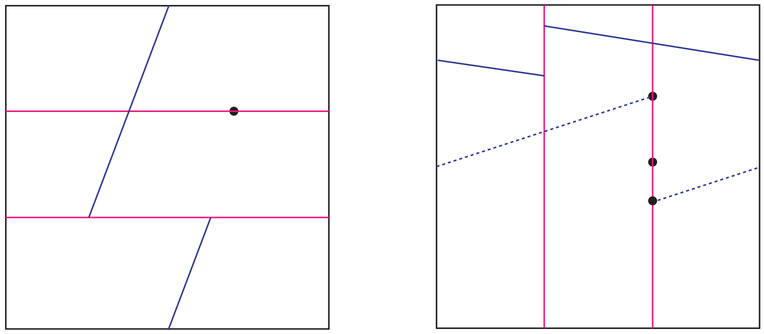

FIGURE 21. The left-hand diagram is T_1 with a bypass arc from the front. After attaching the bypass we get T_2 on the right-hand side, with the corresponding "anti-bypass" to the back, in dashed lines. The solid lines give the bypass attached from the front to give T_3.

the product $R_{\gamma_1}R_{\gamma_2}R_{\gamma_3}R_{\delta_1}^{-2}R_{\delta_2}^{-2}$, where $\gamma_1, \gamma_2, \gamma_3$ are parallel to the two boundary components in the middle and δ_1, δ_2 are the core curves of the middle vertical annuli. Let $\{a_1, a_2, a_3\}$ be the basis, ordered from left to right in Figure 22. (The arcs are on $S'' \times \{-1\}$, where S'' is the page. The α-curves are $\alpha_i = \partial(a_i \times [-1, 1])$.) To determine whether $EH \neq 0$, we apply the Sarkar-Wang algorithm [SW] (also see [Pl]) to isotop β_i so that the only regions of $\Sigma - \cup_i \alpha_i - \cup_i \beta_i$ which do not intersect the suture are quadrilaterals. (Such a diagram we call *combinatorial*.) Let G (in black) be the trivalent graph in the bottom diagram of Figure 22. By isotoping β_3 across G, the Heegaard diagram becomes combinatorial. Now label the intersections $\alpha_1 \cap \beta$ as $x_1, \dots, x_{14}, x_{15} = x_1$ from top to bottom, $\alpha_2 \cap \beta$ as $y_1, \dots, y_9, y_{10} = y_1$ from left to right, and $\alpha_3 \cap \beta$ as $z_1, \dots, z_4 = z_1$ from left to right. Observe that each intersection of the graph with α_i corresponds to two consecutive intersections with the new β_3.

We now directly prove that $EH \neq 0$. If $c_1c_2c_3c_4$ is a quadrilateral which avoids the suture, and $\partial(c_1, c_3, \mathbf{y}) = (c_2, c_4, \mathbf{y}) + \dots$ (for some $\mathbf{y}$), then we say that c_1, c_3 are the "from" corners of the quadrilateral and c_2, c_4 are the "to" corners of the quadrilateral. One readily calculates that the only quadrilateral where the "to" corners are in $\{x_1, y_1, z_1\}$ is $y_1y_2z_1z_2$. Now,

$$\partial(x_1, y_2, z_2) = (x_1, y_1, z_1) + (x_1, y_3, z_2),$$

where the second term comes from a bigon. Next, the only quadrilateral where the "to" corners are in $\{x_1, y_3, z_2\}$ is $x_1x_2y_3y_4$. We compute that

$$\partial(x_2, y_4, z_2) = (x_1, y_3, z_2) + (x_3, y_4, z_2).$$

Continuing, the only quadrilateral with "to" corners in $\{x_3, y_4, z_2\}$ is $x_3x_4z_3z_2$. We have

$$\partial(x_4, y_4, z_3) = (x_3, y_4, z_2) + (x_7, y_1, z_3).$$

There is nothing else with "to" corners in $\{x_7, y_1, z_3\}$. Hence $EH \neq 0$.

(c) Now consider $M''' = T^2 \times [0, 4]$, where $\#\Gamma_{T_0} = \#\Gamma_{T_4} = 2$, $\text{slope}(\Gamma_{T_0}) = \text{slope}(\Gamma_{T_4}) = 0$, and the contact structure ξ''' has "torsion" $\approx 2\pi$. (This is the contact structure described in the first paragraph of Example 6.) The page S''' is obtained from the page S'' for M'' by attaching an annulus along the right-hand boundary in Figure 22. (We can draw the

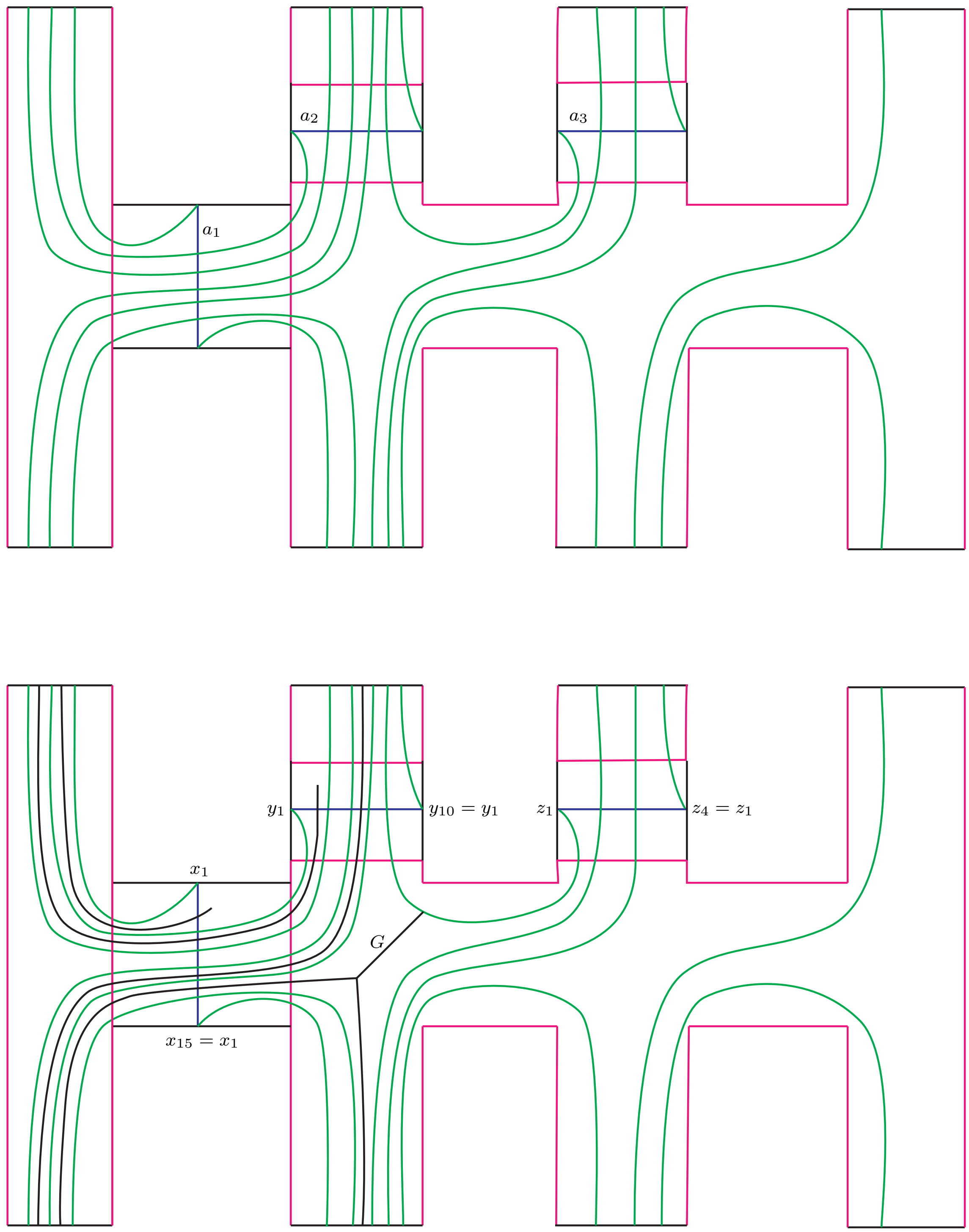

FIGURE 22.

annulus to be vertical and parallel to the previous annuli which were successively attached, starting with S.) Then the monodromy $h''' = R_{\gamma_1} R_{\gamma_2} R_{\gamma_3} R_{\gamma_4} R_{\delta_1}^{-2} R_{\delta_2}^{-2} R_{\delta_3}^{-2}$, where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$

are parallel to the three boundary components in the middle and $\delta_1, \delta_2, \delta_3$ are the core curves of the middle vertical annuli. One can probably directly show that $EH(M''', \Gamma''', \xi''') \neq 0$ with some patience, but instead we invoke Theorem 4.5 and observe that (M''', ξ''') can be embedded in the unique Stein fillable contact structure on T^3, which has nonvanishing contact invariant.

(d) Recall that a contact manifold (M, ξ) has *$n\pi$-torsion* with n a positive integer if it admits an embedding $(T^2 \times [0,1], \eta_{n\pi}) \hookrightarrow (M, \xi)$, where (x, y, t) are coordinates on $T^2 \times [0,1] \simeq \mathbb{R}^2/\mathbb{Z}^2 \times [0,1]$ and $\eta_{n\pi} = \ker(\cos(n\pi t)dx - \sin(n\pi t)dy)$. In a subsequent paper [GHV], Ghiggini, the first author, and Van Horn-Morris prove that if (M, ξ) has torsion $\geq 2\pi$ with M closed, then $EH(M, \xi) = 0$ when $\mathbb{Z}$-coefficients are used.

6. Relationship with sutured manifold decompositions

A sutured manifold decomposition $(M, \Gamma) \overset{T}{\rightsquigarrow} (M', \Gamma')$ is *well-groomed* if for every component R of $\partial M - \Gamma$, $T \cap R$ is a union of parallel oriented nonseparating simple closed curves if R is nonplanar and arcs if R is planar. According to [HKM3], to each well-groomed sutured manifold decomposition there is a corresponding convex decomposition $(M, \Gamma) \overset{(T,\Gamma_T)}{\rightsquigarrow} (M', \Gamma')$, where T is a convex surface with Legendrian boundary and Γ_T is a dividing set which is ∂-parallel, i.e., each component of Γ_T cuts off a half-disk which intersects no other component of Γ_T.

In [Ju2], Juhász proved the following theorem:

Theorem 6.1 (Juhász)**.** *Let $(M, \Gamma) \overset{T}{\rightsquigarrow} (M', \Gamma')$ be a well-groomed sutured manifold decomposition. Then $SFH(-M', -\Gamma')$ is a direct summand of $SFH(-M, -\Gamma)$.*

More precisely, $SFH(-M', -\Gamma')$ is the direct sum $\oplus_{\mathfrak{s}} SFH(-M, -\Gamma, \mathfrak{s})$, where $\mathfrak{s}$ ranges over all relative Spin^c-structures which evaluate maximally on T. (The are called "outer" in [Ju2].)

In this section we give an alternate proof from the contact-topological perspective. We remark that our proof does not require the use of the Sarkar-Wang algorithm. In particular, our proof will imply the following:

Theorem 6.2. *Let (M, Γ, ξ) be the contact structure obtained from (M', Γ', ξ') by gluing along a ∂-parallel (T, Γ_T). Under the inclusion of $SFH(-M', -\Gamma')$ in $SFH(-M, -\Gamma)$ as a direct summand, $EH(M', \Gamma', \xi')$ is mapped to $EH(M, \Gamma, \xi)$.*

Corollary 6.3. *$EH(M', \Gamma', \xi') \neq 0$ if and only if $EH(M, \Gamma, \xi) \neq 0$.*

Proof. The "if" direction is given by Theorem 4.5. The "only if" direction follows from Theorem 6.2. □

The corollary is a gluing theorem for tight contact structures which are glued along a ∂-parallel dividing set, and does not require any universally tight condition which was needed for its predecessors, e.g., Colin's gluing theorem [Co1].

Proof. Let T be a convex surface with Legendrian boundary and Γ_T be its dividing set, which we assume is ∂-parallel. Let (M, Γ, ξ) be a contact structure obtained from (M', Γ', ξ') by

gluing along (T, Γ_T). Suppose without loss of generality that T is oriented so that the disks cut off by the ∂-parallel arcs are negative regions of $T - \Gamma_T$. Let $L \subset T$ be a Legendrian skeleton of the positive region, with endpoints on Γ. We observe that there exist compressing disks D_i in $M - N(L)$ with $\#(\Gamma_{\partial(M-N(L))} \cap \partial D_i) = 2$ so that splitting $M - N(L)$ along the D_i gives (M', Γ'). (The disks are basically the disks in T cut off by L.) See the left-hand side of Figure 23.

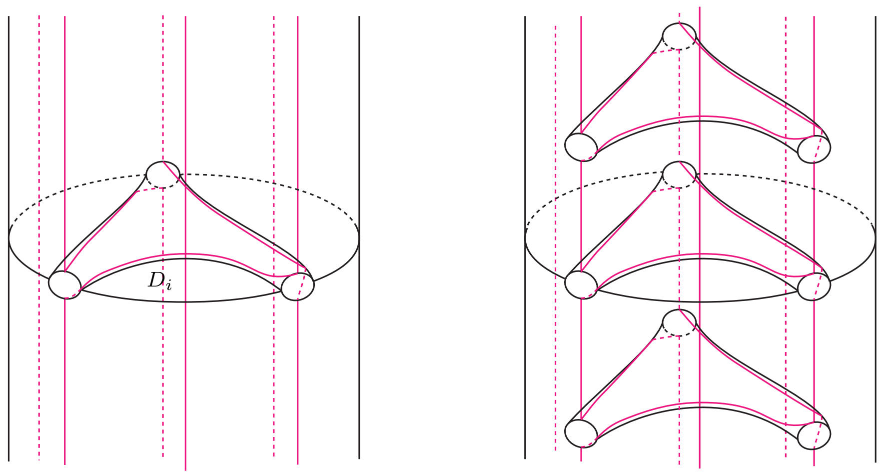

FIGURE 23. The left-hand diagram shows $N(L)$, and the right-hand diagram shows $N(L \cup L_{-\varepsilon} \cup L_\varepsilon)$.

Next we extend L to a Legendrian skeleton K for (M, Γ, ξ), which satisfies the conditions of Theorem 1.1. Before doing this, it would be convenient to "protect" T and L as follows. Since T is convex, there exists a $[-1, 1]$-invariant neighborhood $T \times [-1, 1]$ of $T = T_0$. Let $T_{-\varepsilon}$, T_0 and T_ε be parallel copies of T_0, where $T_t = T \times \{t\}$. Then take parallel Legendrian graphs $L_{i\varepsilon}$, $i = -1, 0, 1$, on $T_{i\varepsilon}$, and thicken to obtain $N(L_{i\varepsilon})$. Here $L = L_0$. Now, apply the technique of Theorem 1.1 to extend $L_{-\varepsilon} \cup L_0 \cup L_\varepsilon$ to K. We may assume that the extension does not intersect the region between $T_{-\varepsilon}$ and T_ε and is disjoint from $L_{-\varepsilon} \cup L_0 \cup L_\varepsilon$. (This is because we can use the compressing disks D_i, together with their translates corresponding to $T_{\pm\varepsilon}$.)

Let $(S, R_+(\Gamma), h)$ be the partial open book decomposition of (M, Γ, ξ) corresponding to the decomposition into $N(K)$ and $M - N(K)$. The three connected components of $S - \overline{R_+(\Gamma)}$ corresponding to $L_{i\varepsilon}$ will be called $Q_{i\varepsilon}$. (Of course there may be other connected components.) If a is a properly embedded arc on $Q = Q_0$ (with endpoints away from Γ), let $a_{i\varepsilon}$ be the corresponding copy on $Q_{i\varepsilon}$. Then $h(a)$ is a concatenation $a' a_\varepsilon a''$, where a' and a'' are arcs which switch levels $Q \leftrightarrow Q_\varepsilon$ as given in Figure 24. The proof that $h(a)$ is indeed as described is similar to the computation of Example 3 in Section 5.

Next, a page of the partial open book decomposition $(S', R_+(\Gamma'), h')$ of (M', Γ', ξ') is given in Figure 25. In particular, all of Q becomes part of the new $R_+(\Gamma')$ and we cut $R_+(\Gamma) \subset S$ along the arcs d_i that correspond to the D_i. (If the d_i are the dotted blue arcs in Figure 24, then $D_i = d_i \times [-1, 1]$.)

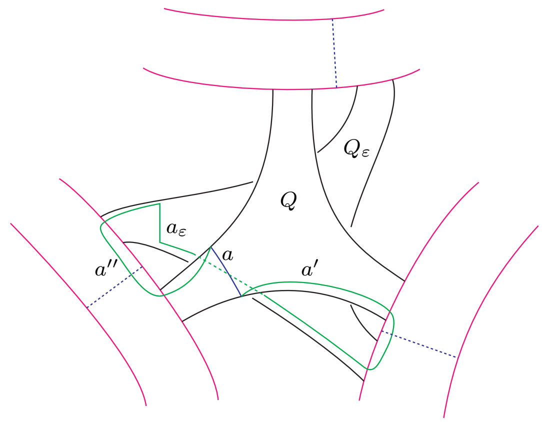

FIGURE 24.

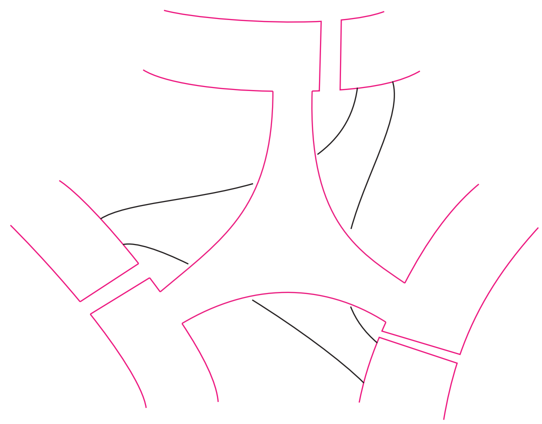

FIGURE 25.

Case 1: Q has no genus. Suppose Q has no genus and hence is a $2k$-gon, where k of the edges are subarcs $\gamma_0, \dots, \gamma_{k-1}$ of Γ and k of the edges are subarcs of ∂S. Assume the γ_i are in counterclockwise order about ∂Q. Consider a basis for Q consisting of $k-1$ arcs $a_1, \dots, a_{k-1}$, where a_i is parallel to γ_i. Let b_i be the pushoffs of a_i satisfying (1), (2) and (3) of Section 2 and let x_i be the intersection of a_i and b_i on $Q \times \{1\}$. Also let $\alpha_i = \partial(a_i \times [-1,1])$ and the $\beta_i = (b_i \times \{1\}) \cup (h(b_i) \times \{-1\})$, viewed on $S \times [-1,1]/_\sim$. (This $[-1,1]$-coordinate is different from the one used previously for the invariant neighborhood of T.) Complete the α_i and β_i into a compatible Heegaard decomposition for (M, Γ) by adding α'_i and β'_i. Hence $\alpha = (\alpha_1, \dots, \alpha_{k-1}, \alpha'_1, \dots, \alpha'_l)$ and $\beta = (\beta_1, \dots, \beta_{k-1}, \beta'_1, \dots, \beta'_l)$.

Claim. *The only $(k+l-1)$-tuples $\mathbf{y}$ of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ whose corresponding first Chern class $c_1(\mathfrak{s}_\mathbf{y})$ evaluates maximally on T have the form $(y_1, \dots, y_{k-1}, y'_1, \dots, y'_l)$, where y_i is an intersection of some α_{j_1} and β_{j_2}, and y'_i is an intersection of some α'_{j_1} and β'_{j_2}.*

Proof. We describe T as it sits inside the partial open book $(S, R_+(\Gamma), h)$. First isotop the d_i above to d'_i as given in Figure 26. Let T' be the union of the $d'_i \times [-1,1]$ and the region $Q' \times \{1\} \subset Q \times \{1\}$ bounded by the $d'_i \times \{1\}$ and the γ_i. Then isotop T' to obtain T so

that $T \cap \Sigma$ is the union of (the shaded region) $\times \{-1\}$ and $Q \times \{1\}$. Here Σ is the Heegaard surface for $(S, R_+(\Gamma), h)$. Observe that the gradient flow line corresponding to $x_i \in \alpha_i \cap \beta_i$

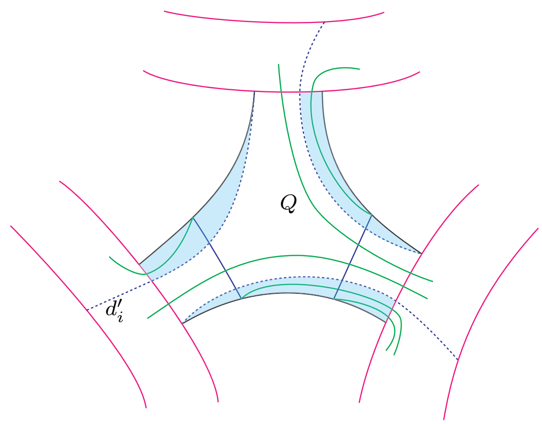

FIGURE 26.

intersects T. The same also holds for any $y_i \in \alpha_{j_1} \cap \beta_{j_2}$, since $h(b_{j_2}) \cap Q$ is contained in the shaded region. On the other hand, the gradient flow line for any intersection point of α_{j_1} and β'_{j_2} does not intersect T, as β'_{j_2} does not enter $Q \times \{1\}$ or the shaded region due to the "protection" afforded by the $Q_{i\varepsilon}$. (The only β'_{j_2} that enter $Q \times \{-1\}$ come from $Q_{-\varepsilon}$. In that case we see in Figure 26 that they are represented by the green arcs outside the shaded region.) Therefore, in order to maximize $\langle c_1(\mathfrak{s}_{\mathbf{y}}), T\rangle$, the intersection point on α_i must lie on β_j. This forces α_i to be paired with β_j and α'_i to be paired with β'_j. □

Still assuming that Q has no genus, consider $\mathbf{y}$ as in the above claim. The only β_i which intersects α_1 is β_1, and their sole intersection is x_1. Hence x_1 occurs in $\mathbf{y}$ — this uses up α_1 and β_1. Next, the only β_i besides β_1 which intersects α_2 is β_2. Continuing in this manner, $\{x_1, \dots, x_{k-1}\} \subset \mathbf{y}$. Hence, the inclusion

$$\Phi : CF(\{\beta'_1, \dots, \beta'_l\}, \{\alpha'_1, \dots, \alpha'_l\}) \to CF(\beta, \alpha),$$

$$(y'_1, \dots, y'_l) \mapsto (x_1, \dots, x_{k-1}, y'_1, \dots, y'_l),$$

is as a direct summand of $CF(\beta, \alpha)$. Moreover, when counting holomorphic disks from $\mathbf{y}' = (x_1, \dots, x_{k-1}, y'_1, \dots, y'_l)$ to $\mathbf{y}'' = (x_1, \dots, x_{k-1}, y''_1, \dots, y''_l)$, the positioning of Γ and the x_i implies that no subarcs of α_i and β_i can be used as the boundary of a nontrivial holomorphic map. Hence holomorphic disks from $\mathbf{y}'$ to $\mathbf{y}''$ are in 1-1 correspondence with holomorphic disks from $(y'_1, \dots, y'_l)$ to $(y''_1, \dots, y''_l)$. This implies that Φ is a chain map. The homology of $CF(\{\beta'_1, \dots, \beta'_l\}, \{\alpha'_1, \dots, \alpha'_l\})$ is $SFH(-M', -\Gamma')$, and image under the injection Φ is the part of $SFH(-M, -\Gamma)$ which is outer. It is also easy to see that $EH(M', \Gamma', \xi')$ is mapped to $EH(M, \Gamma, \xi)$ under Φ.

Case 2: Q has genus. In general, Q has genus g and there are k attaching arcs along Γ. For a general picture, we can think of Figure 24 with added handles. The basis for Q can be chosen to consist of the same arcs $a_1, \dots, a_{k-1}$ parallel to $\gamma_1, \dots, \gamma_{k-1}$ as in Figure 24, plus additional arcs a_{k+2j-2}, a_{k+2j-1}, with one pair for each handle $j = 1, \dots, g$. These additional

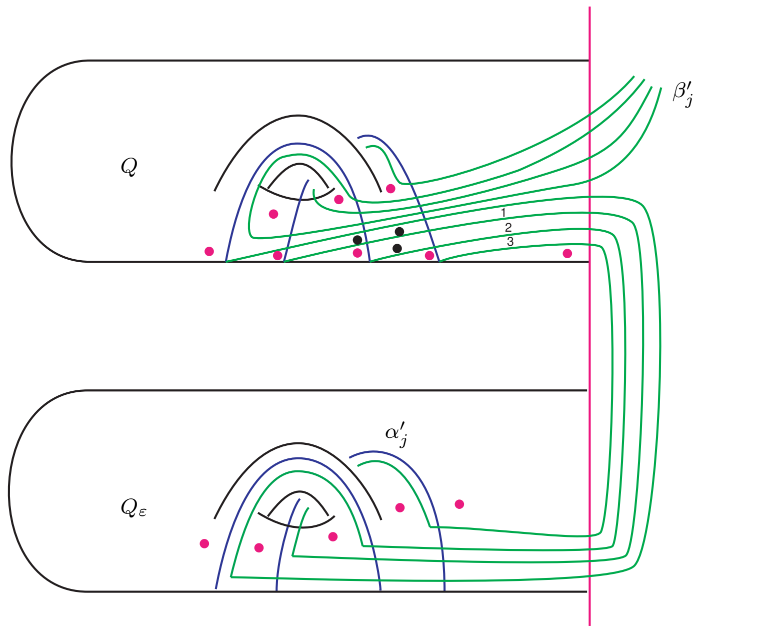

FIGURE 27.

arcs can be taken to end on the subarc τ of ∂S between γ_{k-1} and γ_0, where γ_{k-1}, τ,γ_0 are in counterclockwise order about ∂Q. Figure 27 depicts the case when $k = 1, g = 1$.

Let $a_1, \dots, a_m$, $m = k+2g-1$, be a basis for Q, and b_i, x_i, α_i, β_i, $i = 1, \dots, m$, and α'_j, β'_j, $j = 1, \dots, l$, be as before. The curves that nontrivially intersect $Q \times \{-1\}$ are α_i, β_i, and curves β'_j of the form $((b_i)_{-\varepsilon} \times \{1\}) \cup (h((b_i)_{-\varepsilon}) \times \{-1\})$. Here $(b_i)_{-\varepsilon}$ is an arc on $Q_{-\varepsilon}$ which is parallel to $b_i \subset Q$. Moreover, the curves α'_j and β'_j which pass through $Q_\varepsilon \times \{-1\}$ can be taken to be $\partial((a_i)_\varepsilon \times [-1, 1])$ and $((b_i)_\varepsilon \times \{1\}) \cup (h((b_i)_\varepsilon) \times \{-1\})$, respectively. This is due to the protective layer between $T_{-\varepsilon}$ and T_ε. See Figure 27. We can represent T as before, as the union of (the shaded region) $\times \{-1\}$ and $Q \times \{1\}$, and prove the analogous Claim by using this description of T to show that the α_{j_1} and β_{j_2} must be paired to maximize the first Chern class on T. Hence we only consider $\mathbf{y} = (y_1, \dots, y_m, y'_1, \dots, y'_l)$, where y_i is an intersection of some α_{j_1} and β_{j_2}, and y'_i is an intersection of some α'_{j_1} and β'_{j_2}. As before, if $k > 1$, then we argue that the only β_i which intersects α_1 is β_1, the only β_i besides β_1 which intersects α_2 is β_2, etc., and that $\{x_1, \dots, x_{k-1}\} \subset \mathbf{y}$.

Since the subset $\{x_1, \dots, x_{k-1}\}$ has no effect on what follows, we assume that $k = 1$. Moreover, for simplicity, we assume that $g = 1$. (The higher genus case is not much more difficult.) Consider the pair of curves α_1, α_2 (corresponding to the handle) and the corresponding β curves. Let x_1, v_1 be the intersection points of $\alpha_1 \cap \beta_1$, v_2 be the intersection point of $\alpha_1 \cap \beta_2$, w_1, w_3 be the intersection points of $\alpha_2 \cap \beta_1$, and x_2, w_2 be the intersection points of $\alpha_2 \cap \beta_2$. See Figure 28. Hence the summand $W \subset CF(\beta, \alpha)$ corresponding to the generators which evaluate maximally on T is generated by $(y_1, y_2, \mathbf{y}')$, where (y_1, y_2) is one of

$$(x_1, x_2), (x_1, w_2), (v_1, x_2), (v_1, w_2), (v_2, w_1), (v_2, w_3),$$

and $\mathbf{y}'$ is of the type $(y'_1, \dots, y'_l)$.

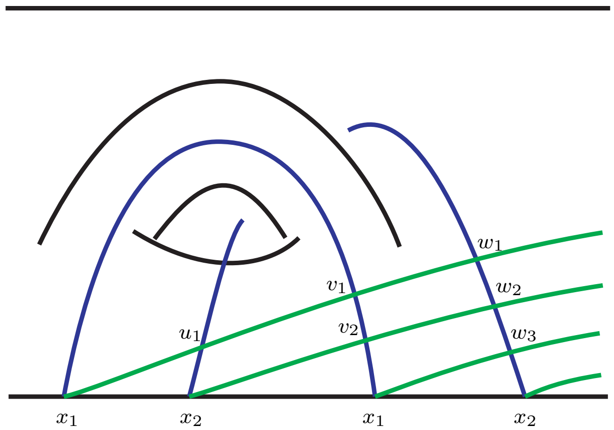

FIGURE 28.

Now we examine the holomorphic disks in $Sym^{m+l}(\Sigma)$ that avoid $\partial\Sigma \times Sym^{m+l-1}(\Sigma)$, in order to compute the boundary maps. The realization of the holomorphic disk as a holomorphic map $\overline{u}$ to Σ cannot have "mixed" boundary components, i.e., each boundary component must consist solely of subarcs of α_i and β_j, or consist solely of subarcs of α_i' and β_j', since all generators of W consist of intersections of α_j and β_k, and intersections of α_j' and β_k' only. Consider the regions of $\Sigma - \cup_i\alpha_i - \cup_i\beta_i - \cup_j\alpha_j' - \cup_j\beta_j'$ which nontrivially intersect Γ — they are indicated by red dots in Figure 27. As a consequence, all the regions in $Q \times \{-1\}$ besides the regions with black dots and the regions 1, 2, and 3, have multiplicity zero. Now, regions 1, 2, and 3 have α_j' curves on the boundary. Hence, if the multiplicity is nonzero, the image of $\overline{u}$ must extend across those curves, in the end engulfing the red-dotted regions in $Q_\varepsilon \times \{-1\}$. Therefore, the multiplicities of regions 1, 2, and 3 are forced to be zero.

The following are the only possible holomorphic disks:

(1) quadrilaterals in $Q \times \{-1\}$ (the regions with black dots in Figure 27);
(2) holomorphic disks from $\mathbf{y}'$ to $\mathbf{y}''$ that do not enter $Q \times \{-1\}$.

The quadrilaterals give rise to boundary maps:

$$\partial(u_1, v_2) = (x_2, v_1),$$
$$\partial(v_1, w_2) = (v_2, w_1),$$
$$\partial(v_2, w_3) = (x_1, w_2),$$
$$\partial(x_1, x_2) = \partial(x_2, v_1) = \partial(v_2, w_1) = \partial(x_1, w_2) = 0,$$

where we are referring to the Heegaard diagram with the α_i', β_i' erased. If $\mathbf{y_i}$ denotes a linear combination of tuples of type $(y_1', \dots, y_l')$ (by slight abuse of notation), then

$$\partial((x_1, x_2, \mathbf{y_1}) + (u_1, v_2, \mathbf{y_2}) + (x_2, v_1, \mathbf{y_3}) + \dots) = 0$$

implies

$$(x_1, x_2, \partial\mathbf{y_1}) + (x_2, v_1, \mathbf{y_2}) + (u_1, v_2, \partial\mathbf{y_2}) + (x_2, v_1, \partial\mathbf{y_3}) + \dots = 0.$$

Here $\partial\mathbf{y_i}$ refers to the boundary map for the Heegaard diagram with $\alpha_1, \alpha_2, \beta_1, \beta_2$ erased. This implies that $\partial\mathbf{y_1} = \partial\mathbf{y_2} = 0$, and $\partial\mathbf{y_3} = \mathbf{y_2}$. Hence,

$$(x_1, x_2, \mathbf{y_1}) + (u_1, v_2, \mathbf{y_2}) + (x_2, v_1, \mathbf{y_3}) + \dots = (x_1, x_2, \mathbf{y_1}) + \partial(u_1, v_2, \mathbf{y_3}) + \dots,$$

and the inclusion

$$\Phi : CF(\{\beta_1', \dots, \beta_l'\}, \{\alpha_1', \dots, \alpha_l'\}) \to CF(\beta, \alpha),$$
$$(y_1', \dots, y_l') \mapsto (x_1, x_2, y_1', \dots, y_l'),$$

induces an isomorphism of $SFH(-M', -\Gamma')$ onto a direct summand of $SFH(-M, -\Gamma)$, as before. □

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